Highlights on Butler B(n)-groups (in collaboration with Clorinda De Vivo).

In this talk, group = torsionfree Abelian group of finite rank.

Rank 1 groups = additive subgroups of $\mathbb{Q}$; their isomorphism classes = *types*;

Direct sums of rank 1 groups = *completely decomposable (c.d.)* groups.

In his paper of 1967: "A class of torsion-free abelian groups of finite rank", M. Butler proved that *torsionfree quotients of c.d groups* and *pure subgroups of c.d groups* (of finite rank) are in fact the same class, the class of *Butler groups*.

The study of Butler groups uses traditionally, as a basic equivalence, *quasi-isomorphism* (= isomorphism up to finite index) instead of isomorphism. (I will say "*isomorphic*" instead of "quasi-isomorphic").

A Butler *B(n)-group* W is *a torsionfree quotient of a c.d. group* Y:

$W = Y/K_Y,$ where $Y = \langle y_1\rangle_* \oplus \langle y_2\rangle_* \oplus \dots \oplus \langle y_r\rangle_*$,

$K_Y = \langle a_1, a_2, \dots, a_n\rangle_*$ is a pure *rank n* subgroup of Y, $a_\ell = \sum_{j=1}^{r} \alpha_{\ell,j} y_j$ $(\ell = 1, \dots, n)$.

In $W = Y/K_Y$, setting $w_j = y_j + K_Y$ we get $W = \langle w_1\rangle_* + \langle w_2\rangle_* + \dots + \langle w_r\rangle_*$:

- a finite sum of *pure rank 1 subgroups* $\langle w_j\rangle_*$ (w.l.o.g. we may suppose $\langle y_j\rangle_* \simeq \langle w_j\rangle_*$ for all j)
- tied by *n independent linear relations* $(n \le r)$:

$$(*)\qquad \sum_{j=1}^{r} \alpha_{\ell,j} w_j = 0 \quad (\ell = 1, ..., n); \quad \text{set} \quad A = \begin{bmatrix} \alpha_{1,1} & \dots & \alpha_{1,j} & \dots & \alpha_{1,r} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{l,1} & \dots & \alpha_{l,j} & \dots & \alpha_{l,r} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n,1} & \dots & \alpha_{n,j} & \dots & \alpha_{n,r} \end{bmatrix}.$$

The *n* conditions (∗), summarized by the matrix A, yield the **linear setting** for W.

E.g., for n = 0, 1: a B(0)-group is completely decomposable; a B(1)-group has as its one relation w.l.o.g. the diagonal relation $w_1 + w_2 + ... + w_r = 0$, $A = [1, 1, \ldots, 1]$.

Linear combinations of relations are also relations; we call *creel* of W the vector space K generated in $\mathbb{Q}^n$ by the rows of A. W.l.o.g. we exclude matrices A equivalent to block-diagonal ones, since this means W splits trivially.

The **partially ordered setting** for W is given by the choice of the rank 1 groups $\langle w_j \rangle_*$ in the lattice $\mathbb{T}$ of all types; to give the rank 1 groups we give their *types*

$$t(\langle y_j \rangle_*) = t(\langle w_j \rangle_*) = t_W(w_j) = u_j; \qquad u_1, u_2, \ldots u_r \text{ are the } \textit{base types} \text{ of } W.$$

Facts: ● The types of the pure rank 1 subgroups of a Butler group form a finite lattice.

● A finite lattice can be realized as a sub-∧-semilattice of (ℕ, gcd, lcm).

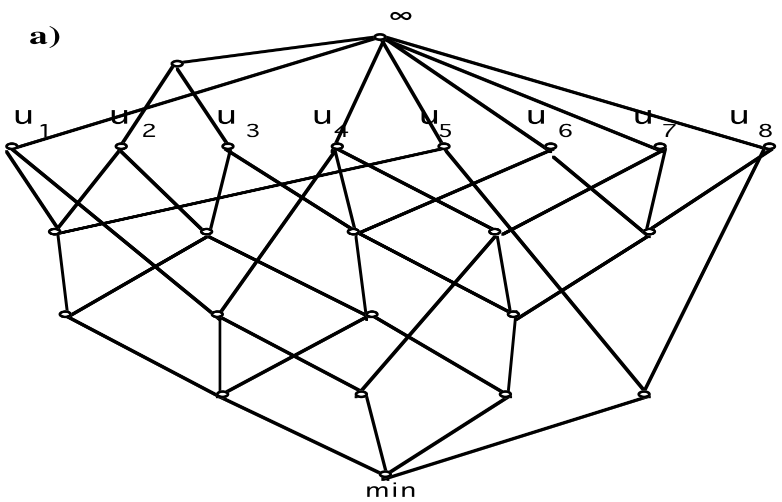
We will outline 4 aspects of our subject.

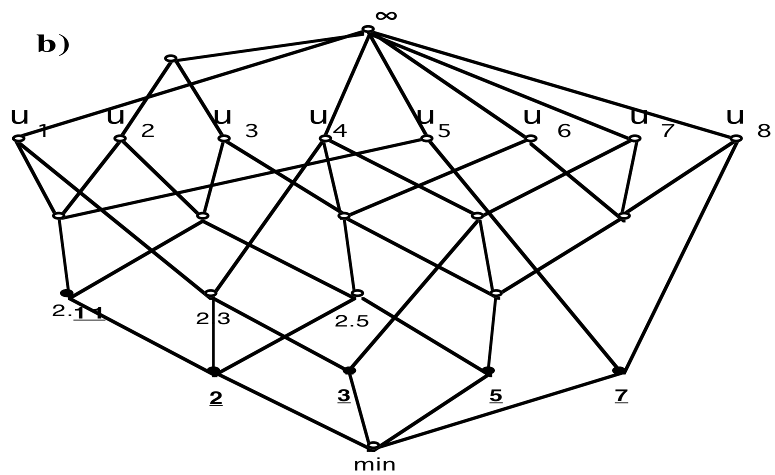
1 : introducing Primes and tents.

Example (DVM4). Let W be a B(1)-group of rank 7 (r = 8), and base types

$u_1 = \infty \;\; \infty \;\; 0 \;\; 0 \;\; \infty \;\; 0 \;\; \infty \;\; 0$ (all zeros)...
$u_2 = \infty \;\; 0 \;\; \infty \;\; 0 \;\; \infty \;\; 0 \;\; \infty \;\; 0$ (all zeros)...
$u_3 = \infty \;\; 0 \;\; \infty \;\; 0 \;\; \infty \;\; \infty \;\; 0 \;\; 0$ (all zeros)...
$u_4 = \infty \;\; \infty \;\; \infty \;\; 0 \;\; 0 \;\; \infty \;\; 0 \;\; 0$ (all zeros)...
$u_5 = \infty \;\; 0 \;\; 0 \;\; \infty \;\; \infty \;\; 0 \;\; \infty \;\; 0$ (all zeros)...
$u_6 = \infty \;\; 0 \;\; \infty \;\; 0 \;\; 0 \;\; \infty \;\; 0 \;\; \infty$ (all zeros)...
$u_7 = 0 \;\; \infty \;\; \infty \;\; 0 \;\; 0 \;\; \infty \;\; 0 \;\; \infty$ (all zeros)...
$u_8 = 0 \;\; 0 \;\; \infty \;\; \infty \;\; 0 \;\; \infty \;\; 0 \;\; \infty$ (all zeros)...

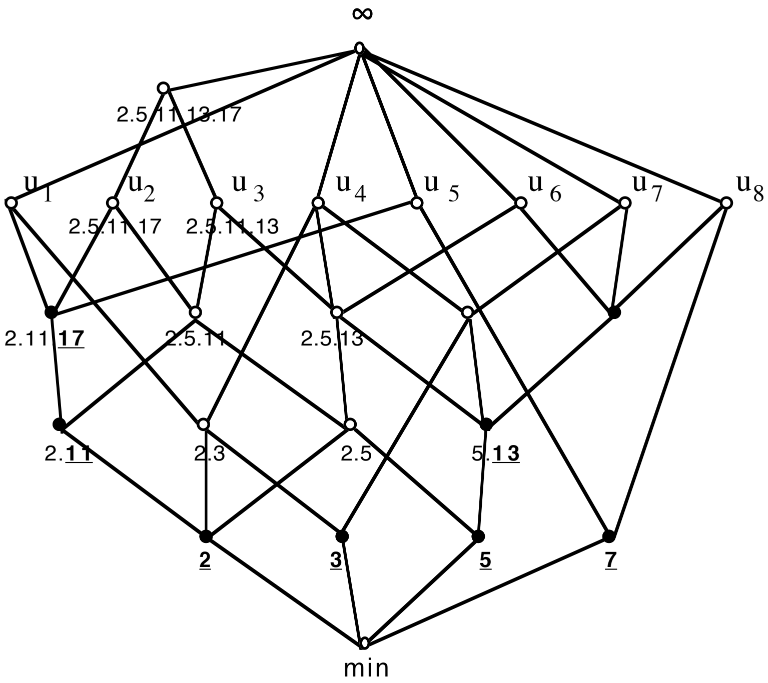
Its typeset is the lattice **a)**;
- start giving Primes (capitalized!) as names to the minimal types, as in **b)**;
- at level 2, give a type the squarefree product of the lower types or, if there is only one lower type, add a new Prime;
- . . . .





By finite induction, each type in the end will receive a finite product of Primes; ∨-irreducible types are marked by new Primes.

In particular, each base type u_j will be given a product of Primes:



u_1	=	2	3	.	.	11	.	17	.
u_2	=	2	.	5	.	11	.	17	.
u_3	=	2	.	5	.	11	13	.	.
u_4	=	2	3	5	.	.	13	.	.
u_5	=	2	.	.	7	11	.	17	.
u_6	=	2	.	5	.	.	13	.	19
u_7	=	.	3	5	.	.	13	.	19
u_8	=	.	.	5	7	.	13	.	19

We have thus represented the *typeset* of W as a *sub-∧-semilattice* of the lattice $\mathbb{N}$ (in fact, of the sublattice of squarefree natural numbers).

What really defines the Prime 2 (e.g.) is the fact that *it divides all base types except for u_7, u_8;* we call $\{7,8\} = F \subseteq J$ the *zero-block* of 2, and set $2 = q_{78} = q_F$.

Thus: *Primes correspond to the subsets of J*.

We get the *tent* of W:

$u_1 =$	q_{78}	q_{2356}	·	·	q_{4678}	·	q_{34678}	·
$u_2 =$	q_{78}	·	q_{15}	·	q_{4678}	·	q_{34678}	·
$u_3 =$	q_{78}	·	q_{15}	·	q_{4678}	q_{125}	·	·
$u_4 =$	q_{78}	q_{2356}	q_{15}	·	·	q_{125}	·	·
$u_5 =$	q_{78}	·	·	$q_{J\setminus\{5,8\}}$	q_{4678}	·	q_{34678}	·
$u_6 =$	q_{78}	·	q_{15}	·	·	q_{125}	·	q_{12345}
$u_7 =$	·	q_{2356}	q_{15}	·	·	q_{125}	·	q_{12345}
$u_8 =$	·	·	q_{15}	$q_{J\setminus\{5,8\}}$	·	q_{125}	·	q_{12345}

2 : splitting B(1)-groups via their tents.

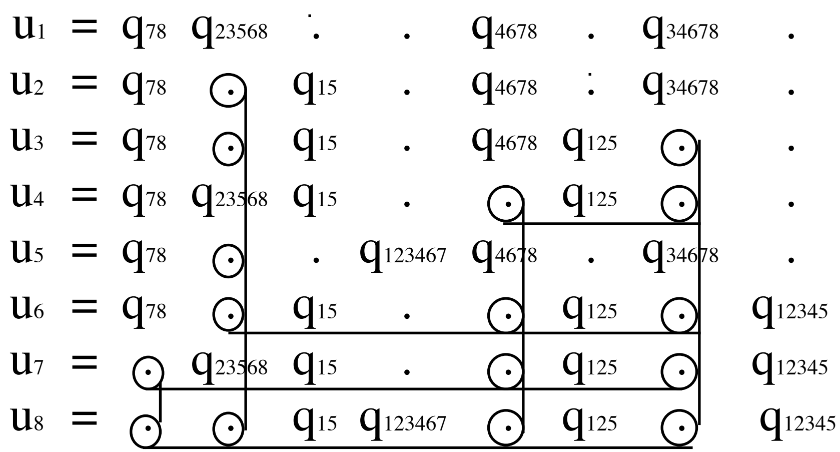
For B(1)-groups, the diagonal relation $w_1 + w_2 + \dots + w_r = 0$ is the same for all groups of the same rank r–1, thus *their structure is determined by the tent*. How?

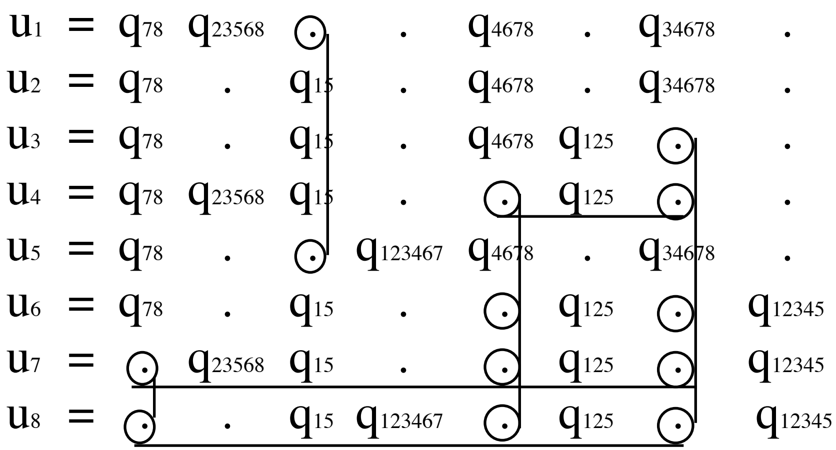
Example. Determine whether the above W is decomposable.

In the next page: 1) start with the first base type u_1. Consider its Primes: $\mathbf{q_{78}}$, $\mathbf{q_{2356}}$, $\mathbf{q_{4678}}$, $\mathbf{q_{34678}}$ and their columns, and connect the dots (circled) horizontally and vertically. Now pull: all rows (except the first) come out together: no conclusion.

2) Go to u_2, consider its Primes: $\mathbf{q_{78}}$, $\mathbf{q_{15}}$, $\mathbf{q_{4678}}$, $\mathbf{q_{34678}}$ and connect their dots (circled) horizontally and vertically. Now pull: the rows come out *in two pieces*: {1,5} and {3,4,6,7,8}.

We say: *the tent of W splits under u_2*. Then **the group splits!**





Theorem (DVM4): *A B(1)-group splits if and only if its tent splits under a base type*. Moreover, the summands can be describd by their tents, and a finite induction yields a complete direct decomposition of W into indecomposables.

What happens with a B(2)-group? Here there are two relations:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_s \end{bmatrix}.$$

In most cases also for B(2)-groups tent-(= order-theoretical-) conditions are sufficient to decide about splitting; but in special cases the linear conditions plays a role:

Example (DVM12). Let the second relation be $\alpha_3 w_3 + \alpha_4 w_4 + \alpha_5 w_5 + \alpha_6 w_6 = 0$,
W the B(2)-group with tent

u_1	=	p_1	p_2	.	.	.	.	.	.
u_2	=	.	.	p_3	p_4	.	.	.	.
u_3	=	.	p_2	.	p_4	s_3	.	.	.
u_4	=	.	p_2	p_3	.	.	s_4	.	.
u_5	=	p_1	.	.	p_4	.	.	s_5	.
u_6	=	p_1	.	p_3	.	.	.	.	s_6

Fact: W splits if and only if $\alpha_3(\alpha_4 - \alpha_6)(\alpha_2 - \alpha_5) - \alpha_4(\alpha_3 - \alpha_5)(\alpha_6 - \alpha_2) = 0$.

The decomposition problem for B(n)-groups is still open.

The other main open problem is that of base-changes, that is, recognizing isomorphism from the base and the creel.

3 : Defining a B(n)-group.

To give a B(n)-group of a given rank we must give the creel K,that is the linear conditions, and the base types, that is the Primes q_F of the tent, i.e. the subsets $F \subseteq J$ of their holes. Can we choose them freely?

This is true for B(0) (= c.d.) groups: e.g. the most ‘general’ tent for a c.d. X of rank 4,

$$X = \langle x_1 \rangle_* \oplus \langle x_2 \rangle_* \oplus \langle x_3 \rangle_* \oplus \langle x_4 \rangle_*, \quad \text{is}$$

u_1	=	·	p_2	p_3	p_4	·	p_{34}	p_{24}	.	p_{23}	.	p_{234}	.	.	.
u_2	=	p_1	·	p_3	p_4	·	p_{34}	.	p_{13}	.	p_{14}	.	p_{134}	.	.
u_3	=	p_1	p_2	·	p_4	p_{12}	.	p_{24}	.	.	p_{14}	.	.	p_{124}	.
u_4	=	p_1	p_2	p_3	·	p_{12}	·	·	p_{13}	p_{23}	.	.	.	.	p_{123}

Instead, whoever worked on B(1)-groups knows that Primes with only one hole are forbidden: given for W the diagonal relation $w_1 + w_2 + \ldots + w_r = 0$, a Prime dividing all but one of the base elements must divide them all.

But the same happens for *any* of the relations of a B(n)-group! If - say - $w_1 + 2w_2 + w_3 = 0$, any Prime dividing - say - w_1 and w_2 must divide w_3: the *supports* of the relations forbid certain Primes.

In fact, *the linear setting A (the creel K) conditions the order-theretical setting*.

Problem: determine the Primes allowed by A.

Call the subset $F \subseteq J$ *regular* if it determines a Prime q_F allowed by A.

Our problem becomes: *determine all regular subsets of J*.

For $n = 0$, a B(0)- (= c.d.) group poses no conditions: its tent has all $2^r - 2$ Primes (we don't write the full and the empty Prime); all subsets of J are regular.

For $n = 1$, the only relation being w.l.o.g. the diagonal relation, the only forbidden Primes are those with only one hole (all subsets of J are regular except for singletons).

In general, determining the regular subsets of J means determining the relations (the elements of K) with *proper support*; the forbidden Primes are those whose zero-block F *pierces* the support of an element of K.

Example. For n = 2, let J = {1, ..., 6}, with relations $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$;

writing e.g. y_{34} for $y_3 + y_4$, we have $K = < y_{12} + y_{34} + y_{56}, y_{34} - y_{56} >$.

The elements of K with proper support are then $y_{34} - y_{56}$, $y_{12} + 2y_{56}$, $y_{12} + 2y_{34}$; e.g. F = {3,4,5} is forbidden, and so is {1,2,3,5}. In fact, we have

<u>Theorem</u> (DVM16): *q_F is a Prime allowed by the creel K of W if and only if F is a union of F_i, where the F_i are minimal dependent* ***sets of columns*** *of A.*

Example. Let us compute the 'maximal' tent of a W with creel K of matrix

$$A = \left[\begin{array}{cc:cc:cc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right].$$

- Sets of a *single* column are not dependent: there is no Prime with just *one* hole.

- Sets F of *two* are dependent if and only if the two colmns are proportional, in which case F is also minimal: $F_1 = \{1,2\}$, $F_2 = \{3,4\}$, $F_3 = \{5,6\}$ will yield all allowed Primes with *two* holes.

- Sets of *three* are always dependent; they are minimal if and only if they don't contain the previous ones: then $F_4 = \{1,3,5\}$, $F_5 = \{1,3,6\}$, $F_6 = \{1,4,5\}$, $F_7 = \{1,4,6\}$, $F_8 = \{2,3,5\}$, $F_9 = \{2,3,6\}$, $F_{10} = \{2,4,5\}$, $F_{11} = \{2,4,6\}$: these will yield all Primes with *three* holes, and here end the minimal Primes.
- sets of *four* are always dependent but not minimal; all of them are unions of minimals, hence all Primes with *four* holes are allowed; so are all Primes with *five* holes:

$u_1=$	.	q_{34}	q_{56}	.	.	.	.	q	q	q	q	....
$u_2=$	.	q_{34}	q_{56}	q_{135}	q	q	q	.	.	.	.	....
$u_3=$	q_{12}	.	q_{56}	.	.	q	q	.	.	q	q	....
$u_4=$	q_{12}	.	q_{56}	q	q	.	.	q	q	.	.	....
$u_5=$	q_{12}	q_{34}	.	.	q	.	q	.	q	.	q	....
$u_6=$	q_{12}	q_{34}	.	q	.	q	.	q	.	q	.	....

The B(n)-groups W whose tent has all the Primes allowed by the linear relations of K are called *total:* W_{tot}.

We have: *Total B(n)-groups are indecomposable!* One such tent for each creel K (Note that each tent stays for $2^{\aleph_0}$ pairwise noncomparable B(n)-groups).

Theorem: the tent of an *arbitrary B(n)-group W with creel K* is obtained from the tent of $W_{tot}(K)$ by *cancelling Primes*; this cancellation *is* the order-theoretical condition defining W.

This result changes the way we consider B(n)-groups: in particular, we may solve a problem for total B(n)-groups - defined just by the linear conditions - and then adapt the solution to all other B(n)-groups with the same creel, as in the next part.

4: B(n)-groups as pure subgroups of c.d. groups.

Starting with W of creel K, we first solve the inclusion problem for $W_{tot}(K)$; when we find a *c.d. pure-container X* of $W_{tot}(K)$, we get one for W by just cancelling from the tent of X the Primes not in W.

The first obstacle to be met in determining X is the *notation* of Primes: q_F of W is denoted by the subset $F \subseteq J$ (index set of the base of W); when $W \leq_* X$, where

$$X = \langle x_1 \rangle_* \oplus \langle x_2 \rangle_* \oplus \ldots \oplus \langle x_m \rangle_*, \qquad \text{with index set } I = \{1, \ldots, m\},$$

we must translate the Prime q_F of W into a p_E of X, where $E \subseteq I$.

Example. Say W_{tot} is *the* total B(1)-group of rank 3 (for n = 1 there is one per rank), with tent

$u_1 =$	.	q_{34}	q_{24}	.	q_{23}	.	q_{234}	.	.	.
$u_2 =$	.	q_{34}	.	q_{13}	.	q_{14}	.	q_{134}	.	.
$u_3 =$	q_{12}	.	q_{24}	.	.	q_{14}	.	.	q_{124}	.
$u_4 =$	q_{12}	.	.	q_{13}	q_{23}	.	.	.	.	q_{123}

We build a c.d. pure-container X of W_{tot}.

Facts: 1. The *rank* of X is the number of minimal Primes of W_{tot}.

2. Each minimal Prime q_{F_i} of W_{tot} corresponds to the minimal Prime p_i of X;

3. Each non-minimal Prime q_F of W_{tot} with $F = \bigcup_{i \in \mathbf{E}} F_i$ corresponds to the Prime $p_{\mathbf{E}}$ of X.

E.g., if $F_1 = \{1,2\}$, $F_2 = \{1,4\}$, $F_3 = \{2,4\}$, $F_4 = \{1,3\}$, $F_5 = \{2,3\}$, $F_6 = \{3,4\}$, then

$$q_{12} \to p_1,\ q_{14} \to p_2,\ q_{24} \to p_3,\ \ldots \text{ while } q_{124} \to p_{123},\ \ldots$$

The tent of X is then

$$\begin{array}{lcccccccccc}
t_1 = & \cdot & p_2 & p_3 & p_4 & p_5 & p_6 & p_{356} & p_{246} & \cdot & \cdot \\
t_2 = & p_1 & \cdot & p_3 & p_4 & p_5 & p_6 & p_{356} & \cdot & p_{145} & \cdot \\
t_3 = & p_1 & p_2 & \cdot & p_4 & p_5 & p_6 & \cdot & p_{246} & p_{145} & \cdot \\
t_4 = & p_1 & p_2 & p_3 & \cdot & p_5 & p_6 & p_{356} & \cdot & \cdot & p_{123} \\
t_5 = & p_1 & p_2 & p_3 & p_4 & \cdot & p_6 & \cdot & p_{246} & \cdot & p_{123} \\
t_6 = & p_1 & p_2 & p_3 & p_4 & p_5 & \cdot & \cdot & \cdot & p_{145} & p_{123}
\end{array}$$

4. The *inclusion matrix* Ω, which will include in X *any* B(1)-group of rank 3, is

$$\Omega = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix}$$

e.g.: $w_1 = x_1 + x_2 + x_4$, where $t_X(w_1) = t_1 \wedge t_2 \wedge t_4 = p_3\, p_5\, p_6\, p_{356} = q_{24}\, q_{34}\, q_{23}\, q_{234} = u_1$.

In fact, *even the inclusion matrix is built from the minimal Primes of* W_{tot}, thus depends on the linear structure of W.

Ω includes W_{tot} purely in X; if now we want a c.d. pure-container of W, it is enough to delete from the tent of X the Primes not in W.

DeVivo-Metelli REFERENCES

DVM1. De Vivo, C and Metelli, C., *B(1)-groups: some counterexamples,* Abelian Groups and Modules, Lecture Notes in Pure and Appl. Math., 182, (1996), 227-232.
DVM2. De Vivo, C. and Metelli, C., *Finite partition lattices and Butler groups,* Comm. Algebra 27 (1999), No.4, 1571-1590.
DVM3. De Vivo, C. and Metelli, C., *Admissible matrices as base changes of B(1)-groups: a realizing algorithm,* Trends in Mathematics,1999, Birkhauser Verlag Basel, 135-147.
DVM4. De Vivo, C. and Metelli, C., *Decomposing B(1)-groups: an algorithm,* Comm.Algebra 30, No.12 (2002), 5621-5637.
DVM5. De Vivo, C. and Metelli, C., *Z2 - linear order-preserving transformations of tents,* Ricerche di Matematica, Vol.LI, Fasc.1°, (2002), 159-184.
DVM6. De Vivo, C. and Metelli, C., *A transvection decomposition in GL(n,2),* Colloquium Mathematicum, Vol.94, (2002), no.1, 51-60.
DVM7. De Vivo, C. and Metelli, C., *A constructive solution to the base change decomposition problem in B(1)-groups,* Proceedings of the Algebra Conference - Venezia 2002, Marcel Dekker, Inc. (2004), 119-132.
DVM8. De Vivo, C. and Metelli, C., *On degenerate B(2)-groups,* Houston Math.J., Volume 32 No.3, 2006.
DVM9. De Vivo,C . and Metelli, C., *On direct sums of B(1)-groups,* Comment. Math. Univ. Carolin., 47 ,2, 2006, 189-202.
DVM10. De Vivo, C. and Metelli, C., *Butler groups splitting over a base element,* Colloquium Mathematicum, Vol. 109, No.2 (2007), 297-305.
DVM11. De Vivo, C. and Metelli, C., *Settings for a study of finite rank Butler groups,* Journal of Algebra, 318 (2007),456-483.

DVM12. De Vivo, C. and Metelli, C., *On direct decomposition of Butler B(2)-groups,* Volume in memoria di A.L.S. Corner, Contibutions to Module Theory; Models, Modules and Abelian Groups,W. De Gruyter 2008, 201 -219.
DVM13. De Vivo, C. and Metelli, C., *On Butler B(2)-groups decomposing over two base elements,* to appear on Comment.Math. Univ. Carolin.).
DVM14. De Vivo, C. and Metelli, C., *On the typeset of a B(2)-group,* (to appear on Houston Math. J.)
DVM15. De Vivo, C. and Metelli, C., *Pure B(n)-subgroups of completely decomposable groups,* Ricerche di Matematica, Volume 60, Vol.2 (2011), 237-248.
DVM16. De Vivo, C. and Metelli, C., *Butler's theorem revisited,* Contemporary Mathematics, "Groups and Model Theory" AMS 2012, Conference proceedings of the international conference on groups and model theory, Muelheim an der Ruhr, 2011 (Droste, Fuchs, Tent, Ziegler, Struengmanneditors).

BDVM. Barioli, F., De Vivo, C. and Metelli, C., *On vector spaces with distinguished subspaces and redundant base,* Linear Algebra and its Applications, 374 (2003), 107-126.

CDVM. Caruso, I., De Vivo, C., Metelli, C., *Partition bases and B(1) - groups,* Lecture Notes in Pure and Applied Mathematics;Abelian Groups, Rings, Modules and Homological Algebra; Editors: Pat Goeters and Overtoun M.G. Jenda, Auburn University, Alabama, USA, CRC, 2006, ISBN 1584885521, 97 GBP (Ginzburg).