Fitting height and character degree graphs

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Universita' dell'Aquila

Ischia Group Theory 2012

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G finite group

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$$Irr(G) = \{ irreducible \ characters \ of \ G \}$$

$$cd(G) = \{\chi(1) : \chi \in Irr(G)\}$$

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 $\rho(G) = \{ \text{primes that divide some degree in } cd(G) \} =$

 $\stackrel{\textit{Ito,Michler}}{=} \pi(G) - \{ p \in \pi(G) : \text{ if } P \in Syl_p(G), \text{ then } P \trianglelefteq G, P' = 1 \}$

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character degree graph $\Gamma(G)$

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vertices = $\rho(G)$

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edges : p and q are adjacent if pq divides some degree in cd(G)

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Definition

complete vertex = vertex adjacent to all the others
complete graph = graph with all vertices complete

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complete vertex = vertex adjacent to all the others *complete graph* = graph with all vertices complete

Notation

F = Fit(G)

$$\Phi = \Phi(G)$$

h(G) = Fitting height of G (if G is solvable)

Manz (1985) $\Gamma(G)$ has at most two connected components

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Manz, Willems, Wolf (1989) $diam(\Gamma(G)) \leq 3$

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Lewis (2001) there exists G having graph with diameter 3
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Lewis (2001) groups with disconnected graph are classified
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Problem

Which assumptions on $\Gamma(G)$ to obtain a bound on h(G)?

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For every integer k there exists H with h(H) = k and $\rho(H) = \{p, q\}$ with $p \nsim_{\Gamma(H)} q$.

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For every integer k there exists H with h(H) = k and $\rho(H) = \{p, q\}$ with $p \not\sim_{\Gamma(H)} q$.

Now $\Gamma(G \times H) = \Gamma(G)$ and $h(G \times H) = k$ is arbitrarily large.

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$\Gamma(G)$ has at least two complete vertices

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h(G) unbounded

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 $\Gamma(G)$ has at most one complete vertex

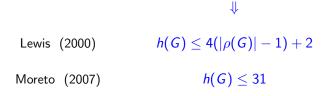
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Lewis (2000) $h(G) \le 4(|\rho(G)| - 1) + 2$

 $\Gamma(G)$ has at most one complete vertex





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 $\Gamma(G)$ has no complete vertices \Downarrow $h(G) \leq 4$

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Example

 $\begin{aligned} & H \simeq GL(2,3) \\ & K : \ \rho(K) = \{p,q\} \text{ with } (pq,6) = 1 \text{ and } p \not\sim_{\Gamma(K)} q \\ & \Gamma(H \times K) \text{ has no complete vertices and } h(H \times K) = 4 \end{aligned}$

 $\Gamma(G)$ has no complete vertices \Downarrow $h(G) \leq 4$

Example

 $H \simeq GL(2,3)$ $K : \rho(K) = \{p,q\} \text{ with } (pq,6) = 1 \text{ and } p \not\sim_{\Gamma(K)} q$ $\Gamma(H \times K) \text{ has no complete vertices and } h(H \times K) = 4$ $\implies the bound is the best possible$

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Theorem 2 (M.Z. 2012)

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Theorem 2 (M.Z. 2012)

 $\Gamma(G)$ has exactly one complete vertex \Downarrow $h(G) \leq 6$

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Conjecture

 $\Gamma(G)$ has exactly one complete vertex $\Rightarrow h(G) \leq 4$

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1. If \Phi = 1 and \pi(F_2/F) \ni p \not\sim q

\Downarrow

\exists! non central minimal normal subgroup M:

C_G(M)/F is a \{p,q\}'- group
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either $h(G/C_G(M)) \leq 2$ or $G/C_G(M) \simeq GL(2,3)$

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 $\xi(G) = \{ \text{primes corresponding to normal non abelian Sylow's} \}$ $\Sigma(G) = \{ \text{primes that are not adjacent to some prime of } \xi(G) \}$

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3. $\Sigma(G)$ -Hall subgroups are abelian

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Following (2), let K be the intersection of all N_{pq} 's obtained by each pair of non adjacent primes of $\pi(G/F)$ and suppose $G/N_{pq} \simeq GL(2,3)$ for any pair;

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 $h(G/K) \leq 2$

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If $p \in \xi(G)$ and $P \in Syl_p(G) \Rightarrow \Gamma(G/P')$ has no complete vertices

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If $G/N_{pq} \simeq GL(2,3) \Rightarrow G/F \simeq GL(2,3) \times H$ with $h(H) \leq 5$

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