Fitting height and character degree graphs

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The character degree graph

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$$\pi(G) = \{primes\ that\ divide\ |G|\}$$

$$\rho(G) = \{primes\ that\ divide\ some\ degree\ in\ cd(G)\} =$$

$$\overset{Ito, Michler}{=} \pi(G) - \{p \in \pi(G) :\ if\ P \in Syl_p(G),\ then\ P \trianglelefteq G, P' = 1\}$$

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Notation

$F = Fit(G)$

$\Phi = \Phi(G)$

$h(G)$= Fitting height of G (if G is solvable)

Properties of $\Gamma(G)$ for solvable groups

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Lewis (2001)
groups with disconnected graph are classified

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$\Gamma(G)$ has at least two complete vertices

$$\Downarrow$$

$h(G)$ unbounded

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Moreto (2007) $h(G) \leq 31$

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Example

$H \simeq GL(2,3)$

$K : \rho(K) = \{p, q\}$ with $(pq, 6) = 1$ and $p \not\sim_{\Gamma(K)} q$

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$\Longrightarrow$ *the bound is the best possible*

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Conjecture

$\Gamma(G)$ *has exactly one complete vertex* $\Rightarrow h(G) \leq 4$

Proof of Theorem 1

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1. If $\Phi = 1$ and $\pi(F_2/F) \ni p \not\sim q$

$$\Downarrow$$

$\exists!$ non central minimal normal subgroup M:

$C_G(M)/F$ is a $\{p, q\}'$- group

either $h(G/C_G(M)) \leq 2$ or $G/C_G(M) \simeq GL(2, 3)$

2. If $p, q \in \pi(G/F)$ with $p \not\sim q$

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$$\exists\ F \leq N_{pq} \trianglelefteq G :$$

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either $h(G/N_{pq}) \leq 2$ or $G/N_{pq} \simeq GL(2,3)$

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3. $\Sigma(G)$-Hall subgroups are abelian

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$$\Rightarrow h(G) \leq 6$$