Torsion Units in the Integral Group Ring $\mathbb{Z}A_n$, $n \leq 10$

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Zassenhause Conjecture

Let G be a finite group and let $V(\mathbb{Z}G)$ denotes the group $\left\{\sum_{g\in G}\alpha_g g \in U(\mathbb{Z}G) : \sum_{g\in G}\alpha_g = 1, \alpha_g \in \mathbb{Z}\right\}$ of normalized units of the integral group ring $\mathbb{Z}G$.

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(ZC) *Every torsion unit u in $V(\mathbb{Z}G)$ is conjugate to an element in G within the rational group algebra $\mathbb{Q}G$; i.e. there exist a group element $g \in G$ and $w \in \mathbb{Q}G$ such that $w^{-1}uw = g$.*

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In parallel to **(ZC)** and as a usefull technique that we have used is the cojecture of W. Kimmerle, which involves the concept of prime graph. For a finite group G, let $pr(G)$ denotes the set of all prime divisors of the order of G. The Gruenberg-Kegel graph (or the prime graph) of G is a $graph(G)$ with vertices labelled by primes from $pr(G)$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group G.

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- In this paper, we confirm **(KC)** for the symmetric group S_7.

In order to state the result, for a group G, let $\mathcal{C} = \{C_1, \ldots, C_{nt}, \ldots\}$ be the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$.

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For any unit $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ of order k, let ν_{nt} denote the partial augmentation $\nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ of u with respect to C_{nt}.

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From Berman's Theorem (see **[2]**), we know that $\nu_1 = \alpha_1 = 0$ and $\nu_c = 0$ for any central element $c \in G$, and that

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we have $\chi(u) = \sum \nu_{nt} \chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}.

Theorem

Let G denote the symmetric group S_7 of degree seven. If u is a torsion unit in $V(\mathbb{Z}G)$ of order $|u|$, and $\mathfrak{PA}(u)$ denotes the tuple

$$(\nu_{2a}, \nu_{2b}, \nu_{2c}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{7a}, \nu_{10a}, \nu_{12a}) \in \mathbb{Z}^{14}$$

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(iii) If $|u| = 2$, the tuple of the partial augmentations $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ of u belongs to the set
$\{(0, -1, 2), (1, -1, 1), (1, 0, 0), (0, 0, 1), (1, 1, -1), (0, 1, 0)\}$ *and* $\nu_{kx} = 0$ *for* $kx \notin \{2a, 2b, 2c\}$.

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Corollary

If G is the symmetric group S_7 of degree seven, then $\pi(G) = \pi(V(G))$.

Preliminaries

For a torsion u in $V(\mathbb{Z}G)$, the **(ZC)** provides that $\chi(u) = \chi(x_i)$ for some $x_i \in G$; and hence an equivalent statement for **(ZC)** was given in the following statement:

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Proposition

(Luthar-Passi & Marciniac-others) If $u \in V(\mathbb{Z}G)$ is a torsion unit of order k. Then u is conjugate to an element $g \in G$ if and only if for each divisor d of k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.

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In fact to establish our investigation, we consider the calculation, by **GAP**, of the indicated numbers $\mu_m(u, \chi)$ in the what follow for each possible order k of a torsion unit u in $V(\mathbb{Z}G)$, taking in account the relations between $|u|$ and the partial augmentations $\nu_i = \varepsilon_{C_i}(u)$ given in the next three Propositions.

Proposition

*(**Hertwick**) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

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(Hertwick & Luthar-Passi) Let either p be a prime divisor of $|G|$ or $p = 0$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k such that k and p are coprime if $p \neq 0$. If z is a primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G then, for every integer m, the number

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{(z^d)/}\{\chi(u^d) z^{-dm}\}$$

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Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

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Since conjugate group elements have same character, then for any normalized unit $u = \sum \alpha_i g_i \in V(\mathbb{Z}S_7)$, its character is $\chi(u) = \sum_{i=1}^{15} \nu_i \chi(x_i)$, where $\nu_i' s (\in \mathbb{Z})$ are partial augmentations $\varepsilon_{C_i}(u)$ of u, and $x_i' s$ are representatives of distinct conjugacy classes C_i in S_7.

If u is torsion in $V(\mathbb{Z}S_7)$ and $|u| = n$, then **(ZC)** provides that $\chi(u) = \chi(x_i)$ for some $x_i \in G$; and hence an equivalent statement for **(ZC)** was given.

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But, by Proposition **4**, the order of each torsion unit divides the exponent 420 of S_7, then it remains to consider only units of orders $14, 15, 20, 21$ and 35. We prove that all units of these orders (except for 20) do not appear in $V(\mathbb{Z}S_7)$.

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Let u be an involution. By Propositions 1 & **2,** we have that $\nu_{2a} + \nu_{2b} + \nu_{2c} = 1$. Applying Proposition **3** to the character χ_2, χ_3 and χ_4, we get the following system

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$$\mu_0(u, \chi_2, *) = \tfrac{1}{2}(\nu_{2a} - \nu_{2b} - \nu_{2c} + 1) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \tfrac{1}{2}(-(\nu_{2a} - \nu_{2b} - \nu_{2c}) + 1) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \tfrac{1}{2}(2\nu_{2a} + 4\nu_{2b} + 6) \geq 0;$$
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\end{aligned}$$

Since all $\mu_i(u, \chi_j, p)$ are non-negative integers, then the only integral solutions are $(\nu_{2a}, \nu_{2b}, \nu_{2c}) \in$
$\{(0, -1, 2), (1, -1, 1), (1, 0, 0), (0, 0, 1), (1, 1, -1), (0, 1, 0)\}$.

Let u be a unit of order 3. By Propositions 1 & 2, we have that $\nu_{3a} + \nu_{3b} = 1$. Applying Proposition **3** to the character χ_2 and χ_3 and from Brauer character tables for $p = 2$ and 3, we get only trivial integer solutions (ν_{3a}, ν_{3b}). So, part (ii) of the Theorem is completed for $|u| = 3$.

Let u be a unit of order 10. By Propositions **1 & 2**, we have

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Case 1. Let $\chi(u^5) = \chi(2a)$. Using Proposition **3,** we get no integral solution.
Case 2. Let $\chi(u^5) = \chi(2b)$. Using Proposition **3,** we get only the following trivial solution (0, 0, 0, 0, 1).
Case 3. Let $\chi(u^5) = \chi(2c)$. Using Proposition **3,** we get no solution.
Case 4. Let $\chi(u^5) = -\chi(2b) + 2\chi(2c)$. Using Proposition **3** we get no solution.
Case 5. Let $\chi(u^5) = \chi(2a) - \chi(2b) + \chi(2c)$. Using Proposition **3,** we get no solution.
Case 6. Let $\chi(u^5) = \chi(2a) + \chi(2b) - \chi(2c)$. Using Proposition **3,** we get no solution.

Let u be a unit of order 10. By Propositions **1 & 2**, we have

$$\nu_{2a} + \nu_{5a} + \nu_{2b} + \nu_{2c} + \nu_{10a} = 1.$$

Since $|u^5| = 2$ for any character χ of S_7 we need to consider six cases, defined by part (iii) of the Theorem. We consider each case separately:
Case 1. Let $\chi(u^5) = \chi(2a)$. Using Proposition **3,** we get no integral solution.
Case 2. Let $\chi(u^5) = \chi(2b)$. Using Proposition **3,** we get only the following trivial solution (0, 0, 0, 0, 1).
Case 3. Let $\chi(u^5) = \chi(2c)$. Using Proposition **3,** we get no solution.
Case 4. Let $\chi(u^5) = -\chi(2b) + 2\chi(2c)$. Using Proposition **3** we get no solution.
Case 5. Let $\chi(u^5) = \chi(2a) - \chi(2b) + \chi(2c)$. Using Proposition **3,** we get no solution.
Case 6. Let $\chi(u^5) = \chi(2a) + \chi(2b) - \chi(2c)$. Using Proposition **3,** we get no solution.
Then, for units of orders 10, there is precisely one conjugacy class with non-zero partial augmentation.

Let u be a unit of order 10. By Propositions **1 & 2**, we have

$$\nu_{2a} + \nu_{5a} + \nu_{2b} + \nu_{2c} + \nu_{10a} = 1.$$

Since $|u^5| = 2$ for any character χ of S_7 we need to consider six cases, defined by part (iii) of the Theorem. We consider each case separately:
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Case 3. Let $\chi(u^5) = \chi(2c)$. Using Proposition **3,** we get no solution.
Case 4. Let $\chi(u^5) = -\chi(2b) + 2\chi(2c)$. Using Proposition **3** we get no solution.
Case 5. Let $\chi(u^5) = \chi(2a) - \chi(2b) + \chi(2c)$. Using Proposition **3,** we get no solution.
Case 6. Let $\chi(u^5) = \chi(2a) + \chi(2b) - \chi(2c)$. Using Proposition **3,** we get no solution.
Then, for units of orders 10, there is precisely one conjugacy class with non-zero partial augmentation.
Therefore, by Proposition **1,** part (ii) of the Theorem is complete.

Let u be a unit of order 14. By Propositions **1&2,** we have

$$\nu_{2a} + \nu_{2b} + \nu_{2c} + \nu_{7a} = 1.$$

Let u be a unit of order 14. By Propositions **1&2,** we have

$$\nu_{2a} + \nu_{2b} + \nu_{2c} + \nu_{7a} = 1.$$

Since $|u^7| = 2$ for any character χ of S_7 we need to consider six cases, defined by part (iii) of the Theorem.

Let u be a unit of order 14. By Propositions **1&2,** we have

$$\nu_{2a} + \nu_{2b} + \nu_{2c} + \nu_{7a} = 1.$$

Since $|u^7| = 2$ for any character χ of S_7 we need to consider six cases, defined by part (iii) of the Theorem.

Case 1. If $\chi(u^7) = \chi(2a)$. Applying Proposition **3** to the character χ_3, we get no solution.

Case 2. If $\chi(u^7) = \chi(2b)$, we get no solution.

Case 3. If $\chi(u^7) = -\chi(2b) + 2\chi(2c)$, we get no solution.

Case 4. If $\chi(u^7) = \chi(2a) - \chi(2b) + \chi(2c)$, we get no solution.

Case 5. If $\chi(u^7) = \chi(2a) + \chi(2b) - \chi(2c)$**,** we get no integral solution.

Case 6. Finally, if $\chi(u^7) = \chi(2c)$. Applying Proposition **3** to the characters χ_2 and χ_3, we get no solution.

Let u be a unit of order 14. By Propositions **1&2,** we have

$$\nu_{2a} + \nu_{2b} + \nu_{2c} + \nu_{7a} = 1.$$

Since $|u^7| = 2$ for any character χ of S_7 we need to consider six cases, defined by part (iii) of the Theorem.
Case 1. If $\chi(u^7) = \chi(2a)$. Applying Proposition **3** to the character χ_3, we get no solution.
Case 2. If $\chi(u^7) = \chi(2b)$, we get no solution.
Case 3. If $\chi(u^7) = -\chi(2b) + 2\chi(2c)$, we get no solution.
Case 4. If $\chi(u^7) = \chi(2a) - \chi(2b) + \chi(2c)$, we get no solution.
Case 5. If $\chi(u^7) = \chi(2a) + \chi(2b) - \chi(2c)$**,** we get no integral solution.
Case 6. Finally, if $\chi(u^7) = \chi(2c)$. Applying Proposition **3** to the characters χ_2 and χ_3, we get no solution.
Therefore there is no unit in $V(\mathbb{Z}S_7)$ of order 14.

Let u be a unit of order 15. By Propositions **1&2** we have $\nu_{3a} + \nu_{3b} + \nu_{5a} = 1$. Since $|u^5| = 3$ for any character χ of G we need to consider two cases.

Let u be a unit of order 15. By Propositions **1&2** we have $\nu_{3a} + \nu_{3b} + \nu_{5a} = 1$. Since $|u^5| = 3$ for any character χ of G we need to consider two cases.

Case 1. Let $\chi(u^5) = \chi(3a)$. Applying Proposition **3** to the character χ_5 of G, we get the system

$$\mu_0(u, \chi_5, *) = \frac{1}{15}(16(\nu_{3a} + \nu_{3b}) + 24) \geq 0;$$
$$\mu_5(u, \chi_5, *) = \frac{1}{15}(-8(\nu_{3a} + \nu_{3b}) + 18) \geq 0,$$

which has no integral solutions (ν_{3a}, ν_{3b}).

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Case 1. Let $\chi(u^5) = \chi(3a)$. Applying Proposition **3** to the character χ_5 of G, we get the system

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$$\mu_5(u, \chi_5, *) = \frac{1}{15}(-8(\nu_{3a} + \nu_{3b}) + 18) \geq 0,$$

which has no integral solutions (ν_{3a}, ν_{3b}).
Case 2. Let $\chi(u^5) = \chi(3b)$. Applying Proposition **3** to the character χ_3 of G, we get the following system

$$\mu_0(u, \chi_3, *) = \frac{1}{15}(8(3\nu_{3a} + \nu_{5a}) + 10) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \frac{1}{15}(-2(3\nu_{3a} + \nu_{5a}) + 5) \geq 0,$$

which has no integral solution.

Let u be a unit of order 15. By Propositions **1&2** we have $\nu_{3a} + \nu_{3b} + \nu_{5a} = 1$. Since $|u^5| = 3$ for any character χ of G we need to consider two cases.

Case 1. Let $\chi(u^5) = \chi(3a)$. Applying Proposition **3** to the character χ_5 of G, we get the system

$$\mu_0(u, \chi_5, *) = \tfrac{1}{15}(16(\nu_{3a} + \nu_{3b}) + 24) \geq 0;$$
$$\mu_5(u, \chi_5, *) = \tfrac{1}{15}(-8(\nu_{3a} + \nu_{3b}) + 18) \geq 0,$$

which has no integral solutions (ν_{3a}, ν_{3b}).

Case 2. Let $\chi(u^5) = \chi(3b)$. Applying Proposition **3** to the character χ_3 of G, we get the following system

$$\mu_0(u, \chi_3, *) = \tfrac{1}{15}(8(3\nu_{3a} + \nu_{5a}) + 10) \geq 0;$$
$$\mu_3(u, \chi_3, *) = \tfrac{1}{15}(-2(3\nu_{3a} + \nu_{5a}) + 5) \geq 0,$$

which has no integral solution.

Hence there is no unit in $V(\mathbb{Z}S_7)$ of order 15.

Let u be a unit of order 21. By Propositions **1&2**, we have

$$\nu_{3a} + \nu_{3b} + \nu_{7a} = 1.$$

Since $|u^7| = 3$ for any character χ of G we need to consider two cases.

Let u be a unit of order 21. By Propositions **1&2**, we have

$$\nu_{3a} + \nu_{3b} + \nu_{7a} = 1.$$

Since $|u^7| = 3$ for any character χ of G we need to consider two cases.
Case 1. If $\chi(u^7) = \chi(3a)$. Applying Proposition **3** to the character χ_3 of G, we get the system

$$\mu_0(u, \chi_3, *) = \frac{1}{21}(12(3\nu_{3a} - \nu_{7a}) + 6) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \frac{1}{21}(-6(3\nu_{3a} - \nu_{7a}) - 3) \geq 0,$$

which has no integral solution for (ν_{3a}, ν_{7a}).

Let u be a unit of order 21. By Propositions **1&2**, we have

$$\nu_{3a} + \nu_{3b} + \nu_{7a} = 1.$$

Since $|u^7| = 3$ for any character χ of G we need to consider two cases.
Case 1. If $\chi(u^7) = \chi(3a)$. Applying Proposition **3** to the character χ_3 of G, we get the system

$$\mu_0(u, \chi_3, *) = \tfrac{1}{21}(12(3\nu_{3a} - \nu_{7a}) + 6) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(-6(3\nu_{3a} - \nu_{7a}) - 3) \geq 0,$$

which has no integral solution for (ν_{3a}, ν_{7a}).
Case 2. If $\chi(u^7) = \chi(3b)$. Applying Proposition **3** to the character χ_3 of G, we get the system

$$\mu_0(u, \chi_3, *) = \tfrac{1}{21}(12(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(-6(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{21}((3\nu_{3a} - \nu_{7a}) + 7) \geq 0,$$

which has no integral solution.

Let u be a unit of order 21. By Propositions **1&2**, we have

$$\nu_{3a} + \nu_{3b} + \nu_{7a} = 1.$$

Since $|u^7| = 3$ for any character χ of G we need to consider two cases.
Case 1. If $\chi(u^7) = \chi(3a)$. Applying Proposition **3** to the character χ_3 of G, we get the system

$$\mu_0(u, \chi_3, *) = \frac{1}{21}(12(3\nu_{3a} - \nu_{7a}) + 6) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \frac{1}{21}(-6(3\nu_{3a} - \nu_{7a}) - 3) \geq 0,$$

which has no integral solution for (ν_{3a}, ν_{7a}).
Case 2. If $\chi(u^7) = \chi(3b)$. Applying Proposition **3** to the character χ_3 of G, we get the system

$$\mu_0(u, \chi_3, *) = \frac{1}{21}(12(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \frac{1}{21}(-6(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \frac{1}{21}((3\nu_{3a} - \nu_{7a}) + 7) \geq 0,$$

which has no integral solution.
Hence there is no unit in $V(\mathbb{Z}S_7)$ of order 21.

Let u be a unit of order 35. By Propositions **1&2,** we have $\nu_{5a} + \nu_{7a} = 1$. Applying Proposition **3** to the character χ_3 of G, we get

$$\mu_0(u, \chi_3, *) = \frac{1}{35}(24(\nu_{5a} - \nu_{7a}) + 4) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \frac{1}{35}(-6(\nu_{5a} - \nu_{7a}) - 1) \geq 0,$$

which leads to a contradiction, and hence there is no unit in $V(\mathbb{Z}S_7)$ of order 35.

Let u be a unit of order 35. By Propositions **1&2,** we have $\nu_{5a} + \nu_{7a} = 1$. Applying Proposition **3** to the character χ_3 of G, we get

$$\mu_0(u, \chi_3, *) = \frac{1}{35}(24(\nu_{5a} - \nu_{7a}) + 4) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \frac{1}{35}(-6(\nu_{5a} - \nu_{7a}) - 1) \geq 0,$$

which leads to a contradiction, and hence there is no unit in $V(\mathbb{Z}S_7)$ of order 35.
Therefore the proof is complete □

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