Supersoluble conditions and transfer control

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Introduction

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Strongly and weakly closed subgroups

Definition

Let T be a subgroup of a finite group G.

a) *A subset W of T is said to be weakly closed in T (with respect to G) if whenever $W^g \subseteq T$ with $g \in G$, we have $W^g = W$.*

b) *A subset S of T is said to be strongly closed in T (with respect to G) if for any element $s \in S$ and for any $g \in G$, $s^g \in T$ implies $s^g \in S$ (that is $S^g \cap T \subseteq S$).*

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Remarks

a) The interesting case is when W and S in the above definitions are subgroups of a Sylow p-subgroup P of a finite group G. In such a case S is strongly closed in P w.r.t. G if and only if

$$(*) S \text{ is strongly closed in } N_G(S) \text{ w.r.t. } G.$$

In some recent papers the subgroups satisfying $(*)$ are called $\mathcal{H}$-subgroups

b) Suppose that $P \in Syl_p(G)$. Then typical examples of strongly closed subgroups in P (w.r.t. G) are

- $P \cap N$ where $N \trianglelefteq G$.
- the subgroups of P when P is cyclic
- the center of P when P is a generalized quaternion group

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Let P be a Sylow p-subgroup of a finite group G and let V be a normal subgroup of P. We say that V controls transfer in G if

$$G/O^p(G) \cong N_G(V)/O^p(N_G(V))$$

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Let G be a finite group and $P \in Syl_p(G)$. Suppose that V is a strongly closed subgroup of P (w.r.t. G) and let

$$(*)\ \Phi(V) = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_n = V$$

a chain where $\Phi(V)$ is strongly closed in P (w.r.t. G), V_i is weakly closed in P (w.r.t. G) and $[V_i : V_{i-1}] = p$ for $i = 1, \ldots, n$. Then we say that $(*)$ is a Φ- chain of V.

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We have the following result

Theorem

[A.L.Gilotti-L.S. 2011] let G *be a finite group and* $P \in Syl_p(G)$*. Let* $V \leq P$ *and suppose that* V *and* $\Phi(V)$ *are strongly closed subgroups in* P *with respect to* G*. Then*

$$G/O^p(G) \cong N/O^p(N)$$

where $N = N_G(V)$.

Proposition

[A.L.Gilotti-L.S. 2012] Let G a finite group and P a Sylow p-subgroup of G. Suppose that V is a strongly closed subgroup of P (w.r.t. G) such that

a) *V possesses a Φ-chain*
b) *P/V is cyclic .*

Then

$$G/O^p(G) \cong N_G(P)/O^p(N_G(P))$$

If moreover p is the smallest prime which divides $|G|$, then G has a normal Sylow p-complement

Remarks

We observe that if we exclude one of the two condition a) or b), then the result is not true.

- Consider for example $G = Sym_4$ and let $P \in Syl_2(G)$. Then $1 \lhd V \lhd P$ is chain in which V (the Klein group) is normal in G and then strongly closed. Moreover P/V is cyclic but V has not a Φ-chain. On the other hand $N_G(P) = P$ but G has not a normal Sylow 2-complement.
- Let $G = SL(2, 7)$ and $P = Q_{16} \in Syl_2(G)$. Let $V = Z(P)$.Then $1 \lhd V \lhd P$ is a chain in which V has a Φ-chain but $P/Z(P)$ is not cyclic. On the other hand $N_G(P) = P$ but G has not a normal Sylow 2-complement.

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- Consider for example $G = Sym_4$ and let $P \in Syl_2(G)$. Then $1 \lhd V \lhd P$ is chain in which V (the Klein group) is normal in G and then strongly closed. Moreover P/V is cyclic but V has not a Φ-chain. On the other hand $N_G(P) = P$ but G has not a normal Sylow 2-complement.
- Let $G = SL(2,7)$ and $P = Q_{16} \in Syl_2(G)$. Let $V = Z(P)$.Then $1 \lhd V \lhd P$ is a chain in which V has a Φ-chain but $P/Z(P)$ is not cyclic. On the other hand $N_G(P) = P$ but G has not a normal Sylow 2-complement.

An other observation is the following. If we substitute the condition $\Phi(V)$ strongly closed with $\Phi(V)$ weakly closed in the definition of a Φ-chain, then the result in the above proposition is not true. For example consider the group $G = PSL(2, 17)$ and let $P \in Syl_2(G)$. Then P is dihedral of order 16. The Frattini subgroup $\Phi(P)$ is cyclic of order 4 and then is weakly closed in P (w.r.t. G). Moreover if V_1 is the subgroup of order 8 then V_1 is weakly closed in P (w.r.t. G) and we have the chain $\Phi(P) = V_o \lhd V_1 \lhd V_2 = P$ with $|V_i/V_{i-1}| = 2$. However $N_G(P) = P$ but G is not 2-nilpotent.

From the above proposition we can deduce the following characterization of supersoluble groups

Proposition

A finite group G is supersoluble if and only if there is a normal subgroup N such that G/N has cyclic Sylow subgroups and for every Sylow subgroup P of G we have that $P \cap N$ has a Φ-chain.

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Strict $p - G$-chain

Definition

Let G be a finite group and $P \in Syl_p(G)$. Let V be a strongly closed subgroup of P (w.r.t. G). We say that

$$1 = V_o \trianglelefteq V_1 \trianglelefteq V_2 \ldots \trianglelefteq V_n = V$$

is a strict $p - G$-chain of V if $V_i, i = 1, \ldots, n$ is weakly closed in P (w.r.t. G) and $|V_i : V_{i-1}| = p, i = 1, \ldots, n$.

We have the following

Proposition

Let G be a finite group, $p \in \pi(G)$ and $P \in Syl_p(G)$. Suppose that V is a strongly closed subgroup of P (w.r.t. G) which possesses a strict $p-G$- chain and $(|N_G(V) : C_G(V)|, p-1) = 1$. Then $V \in Syl_p(V^G)$

Remark

We observe that, considering the chain

$$1 = V_o \trianglelefteq V_1 \trianglelefteq V_2 \ldots \trianglelefteq V_n = V$$

in the above proposition, if we change the condition $|V_i : V_{i-1}| = p$ with $V_i/V_{i-1} \leq Z(V/V_{i-1})$ then the result is not true. For example consider a simple Suzuki group $G = Sz(q)$ and let $P \in Syl_2(G)$. Then $\Omega_1(P) = Z(P)$ and it is strongly closed. However $Z(P)^G = G$