Supersoluble conditions and transfer control

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Strongly and weakly closed subgroups

Definition

Let $T$ be a subgroup of a finite group $G$.

a) A subset $W$ of $T$ is said to be weakly closed in $T$ (with respect to $G$) if whenever $W^g \subseteq T$ with $g \in G$, we have $W^g = W$.

b) A subset $S$ of $T$ is said to be strongly closed in $T$ (with respect to $G$) if for any element $s \in S$ and for any $g \in G$, $s^g \in T$ implies $s^g \in S$ (that is $S^g \cap T \subseteq S$).
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Remarks

a) The interesting case is when $W$ and $S$ in the above definitions are subgroups of a Sylow $p$-subgroup $P$ of a finite group $G$. In such a case $S$ is strongly closed in $P$ w.r.t. $G$ if and only if

\[(*) S \text{ is strongly closed in } N_G(S) \text{ w.r.t. } G.\]

In some recent papers the subgroups satisfying $(*)$ are called $\mathcal{H}$-subgroups

b) Suppose that $P \in Syl_p(G)$. Then typical examples of strongly closed subgroups in $P$ (w.r.t. $G$) are

- $P \cap N$ where $N \trianglelefteq G$.
- the subgroups of $P$ when $P$ is cyclic
- the center of $P$ when $P$ is a generalized quaternion group
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\[(*)\] $S$ is strongly closed in $N_G(S)$ w.r.t. $G$.

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b) Suppose that $P \in Syl_p(G)$. Then typical examples of strongly closed subgroups in $P$ (w.r.t. $G$) are

- $P \cap N$ where $N \unlhd G$.
- the subgroups of $P$ when $P$ is cyclic
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**Definition**

Let $P$ be a Sylow $p$-subgroup of a finite group $G$ and let $V$ be a normal subgroup of $P$. We say that $V$ controls transfer in $G$ if

\[ G/O^p(G) \cong N_G(V)/O^p(N_G(V)) \]

**Definition**

Let $G$ be a finite group and $P \in Syl_p(G)$. Suppose that $V$ is a strongly closed subgroup of $P$ (w.r.t. $G$) and let

\[ (*) \Phi(V) = V_0 \trianglelefteq V_1 \trianglelefteq \cdots \trianglelefteq V_n = V \]

a chain where $\Phi(V)$ is strongly closed in $P$ (w.r.t. $G$), $V_i$ is weakly closed in $P$ (w.r.t. $G$) and $[V_i : V_{i-1}] = p$ for $i = 1, \ldots, n$. Then we say that $(*)$ is a $\Phi$-chain of $V$. 
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We have the following result

**Theorem**

[A.L. Gilotti-L.S. 2011] let $G$ be a finite group and $P \in Syl_p(G)$. Let $V \leq P$ and suppose that $V$ and $\Phi(V)$ are strongly closed subgroups in $P$ with respect to $G$. Then

$$G/O^p(G) \cong N/O^p(N)$$

where $N = N_G(V)$. 
Proposition

[A.L. Gilotti-L.S. 2012] Let $G$ a finite group and $P$ a Sylow $p$-subgroup of $G$. Suppose that $V$ is a strongly closed subgroup of $P$ (w.r.t. $G$) such that

a) $V$ possesses a $\Phi$-chain

b) $P/V$ is cyclic.

Then

$$G/O^p(G) \cong N_G(P)/O^p(N_G(P))$$

If moreover $p$ is the smallest prime which divides $|G|$, then $G$ has a normal Sylow $p$-complement
Remarks

We observe that if we exclude one of the two condition a) or b), then the result is not true.

Consider for example $G = Sym_4$ and let $P \in Syl_2(G)$. Then $1 \triangleleft V \triangleleft P$ is chain in which $V$ (the Klein group) is normal in $G$ and then strongly closed. Moreover $P/V$ is cyclic but $V$ has not a $\Phi$-chain. On the other hand $N_G(P) = P$ but $G$ has not a normal Sylow 2-complement.

Let $G = SL(2, 7)$ and $P = Q_{16} \in Syl_2(G)$. Let $V = Z(P)$. Then $1 \triangleleft V \triangleleft P$ is a chain in which $V$ has a $\Phi$-chain but $P/Z(P)$ is not cyclic. On the other hand $N_G(P) = P$ but $G$ has not a normal Sylow 2-complement.
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- Consider for example $G = Sym_4$ and let $P \in Syl_2(G)$. Then $1 \triangleleft V \triangleleft P$ is a chain in which $V$ (the Klein group) is normal in $G$ and then strongly closed. Moreover $P/V$ is cyclic but $V$ has not a $\Phi$-chain. On the other hand $N_G(P) = P$ but $G$ has not a normal Sylow 2-complement.

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An other observation is the following. If we substitute the condition $\Phi(V)$ strongly closed with $\Phi(V)$ weakly closed in the definition of a $\Phi$-chain, then the result in the above proposition is not true. For example consider the group $G = PSL(2, 17)$ and let $P \in Syl_2(G)$. Then $P$ is dihedral of order 16. The Frattini subgroup $\Phi(P)$ is cyclic of order 4 and then is weakly closed in $P$ (w.r.t. $G$). Moreover if $V_1$ is the subgroup of order 8 then $V_1$ is weakly closed in $P$ (w.r.t. $G$) and we have the chain $\Phi(P) = V_o \triangleleft V_1 \triangleleft V_2 = P$ with $|V_i/V_{i-1}| = 2$. However $N_G(P) = P$ but $G$ is not 2-nilpotent.
From the above proposition we can deduce the following characterization of supersoluble groups

Proposition

A finite group $G$ is supersoluble if and only if there is a normal subgroup $N$ such that $G/N$ has cyclic Sylow subgroups and for every Sylow subgroup $P$ of $G$ we have that $P \cap N$ has a $\Phi$-chain.
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Strict $p - G$-chain

Definition
Let $G$ be a finite group and $P \in Syl_p(G)$. Let $V$ be a strongly closed subgroup of $P$ (w.r.t. $G$). We say that

$$1 = V_0 \unlhd V_1 \unlhd V_2 \ldots \unlhd V_n = V$$

is a strict $p - G$-chain of $V$ if $V_i, i = 1, \ldots, n$ is weakly closed in $P$ (w.r.t. $G$) and $|V_i : V_{i-1}| = p, i = 1, \ldots, n$. 
We have the following

**Proposition**

Let $G$ be a finite group, $p \in \pi(G)$ and $P \in Syl_p(G)$. Suppose that $V$ is a strongly closed subgroup of $P$ (w.r.t. $G$) which possesses a strict $p-G$-chain and $(|N_G(V) : C_G(V)|, p - 1) = 1$. Then $V \in Syl_p(V^G)$.
Remark

We observe that, considering the chain

\[ 1 = V_0 \trianglelefteq V_1 \trianglelefteq V_2 \ldots \trianglelefteq V_n = V \]

in the above proposition, if we change the condition \(|V_i : V_{i-1}| = p\) with \(V_i/V_{i-1} \leq Z(V/V_{i-1})\) then the result is not true. For example consider a simple Suzuki group \(G = Sz(q)\) and let \(P \in Syl_2(G)\). Then \(\Omega_1(P) = Z(P)\) and it is strongly closed. However \(Z(P)^G = G\)