Overgroups of Subsystem Subgroups in Exceptional Groups

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General notations

Φ	–	irreducible reduced root system of type E_6, E_7, E_8, G_2, F_4.
Δ	–	subsystem of Φ (not necessary irreducible)
R	–	commutative associative ring with unity
$G(\Phi, R)$	–	Chevalley group of type Φ
$E(\Delta, R)$	–	Elementary Chavalley group of type Δ
$W(\Delta)$	–	Weyl group of root system Δ considered as a subgroup of $W(\Phi)$

Main conjecture

Let Φ be an irreducible reduced root system, $\Delta \leq \Phi$, satisfying condition (*), R - commutative ring and H be a subgroup of $G(\Phi, R)$ such that

$$E(\Delta, R) \leq H \leq G(\Phi, R),$$

then there exist a unique k-tuple of ideals $(I_\omega)_{\omega=1}^{k}$ in R such that

$$E^*(\Delta, R, I_1, \ldots, I_k) \leq H \leq N_G(E^*(\Delta, R, I_1, \ldots, I_k))$$

k – number of orbits of $W(\Delta)$ on $\Phi \setminus \Delta$
E^* – some subgroup in $G(\Phi, R)$, defined by ideals $I_1, \ldots, I_k$ and containing $E(\Delta, R)$

Description of ideals

Let $E(\Delta, R) \leq H \leq G(\Phi, R)$ For any root $\alpha \in \Phi$
define $I_\alpha := \{\xi \in R | x_\alpha(\xi) \in H\}$

Proposition

1. for any $\alpha \in \Phi$ $\quad I_\alpha$ is an ideal in R
2. for any α, β lying in the same orbit of action of $W(\Delta)$ on $\Phi \setminus \Delta$
   $I_\beta = I_\alpha = I_{[\alpha]}$

Idea of the proof: use Chvalley comutator formula:

$$[x_\alpha(\xi), x_\beta(\zeta)], x_{-\beta}(1)] = x_\alpha(\xi\zeta)$$

(true for any roots α, β, such that $\alpha + \beta$ is a root itself)

Description of subgroup E^*, condition (*)

$k = 1$:

$E^*(\Delta, R, I) = E(\Delta, R)E(\Phi, I)$

$k > 1$:

$E^*(\Delta, R, I_1, \ldots, I_k) = \langle x_\alpha(\xi) | \alpha \in \Phi, \xi \in I_{[\alpha]} \rangle$ - generalization of the notion of the elementary net subgroup.

Remark: difficulty of computations depends on the number of orbits

Condition (*):

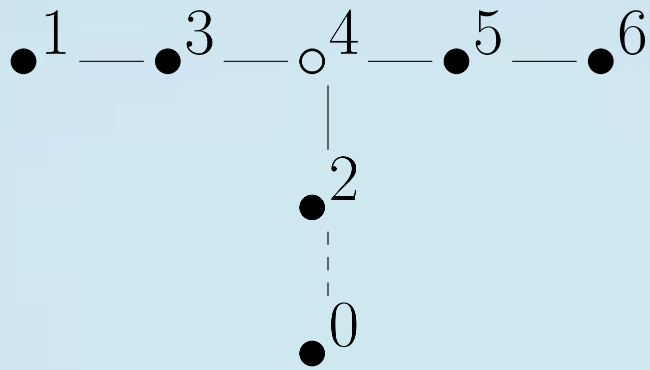
weak: $\Delta^\perp = \emptyset$

strong: for each α in $\Phi \setminus \Delta$ there exists β in an irreducible component of Δ of rank at least 2 such that $\alpha + \beta$ is a root in Φ.

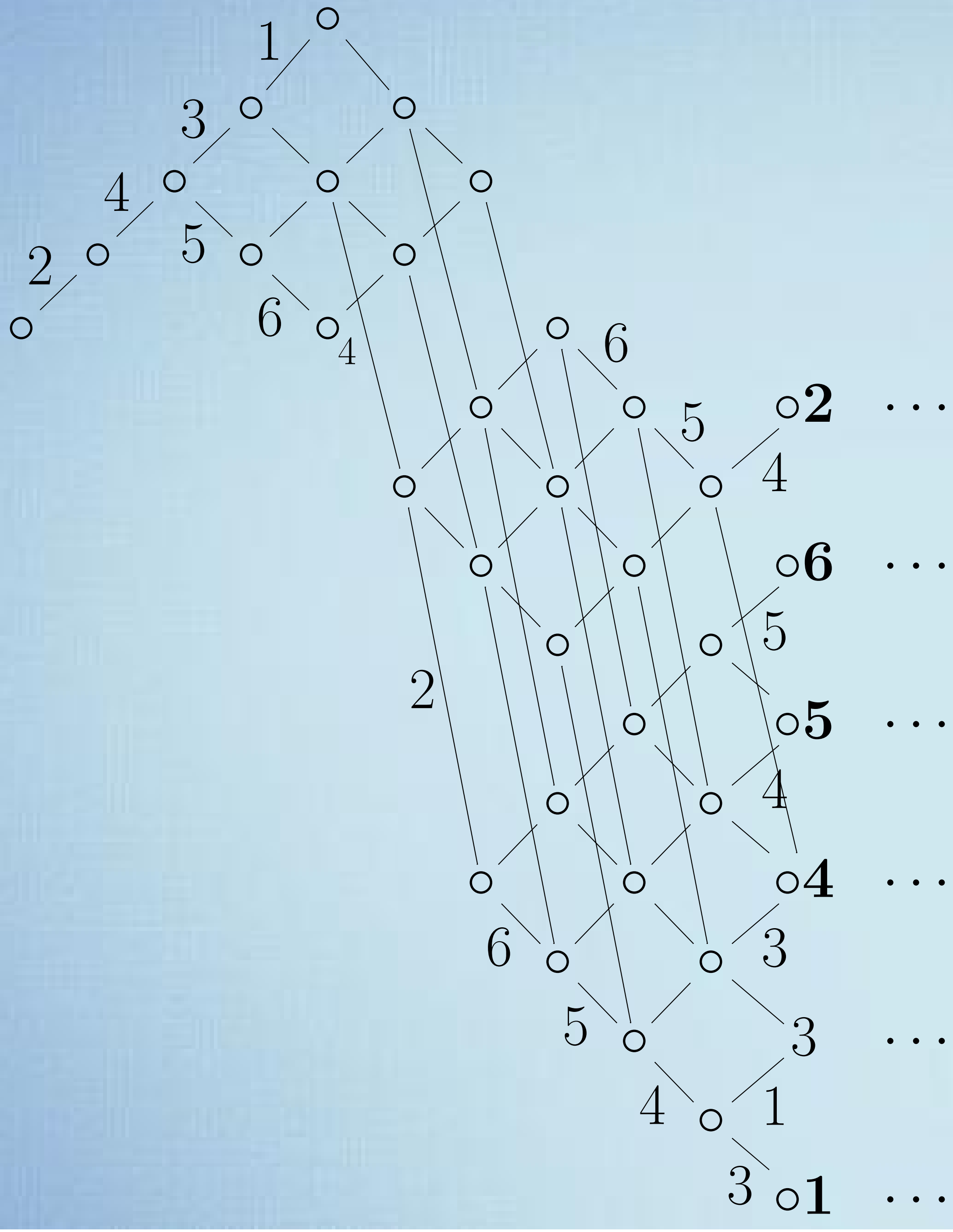
Example:

$\Phi = E_6$

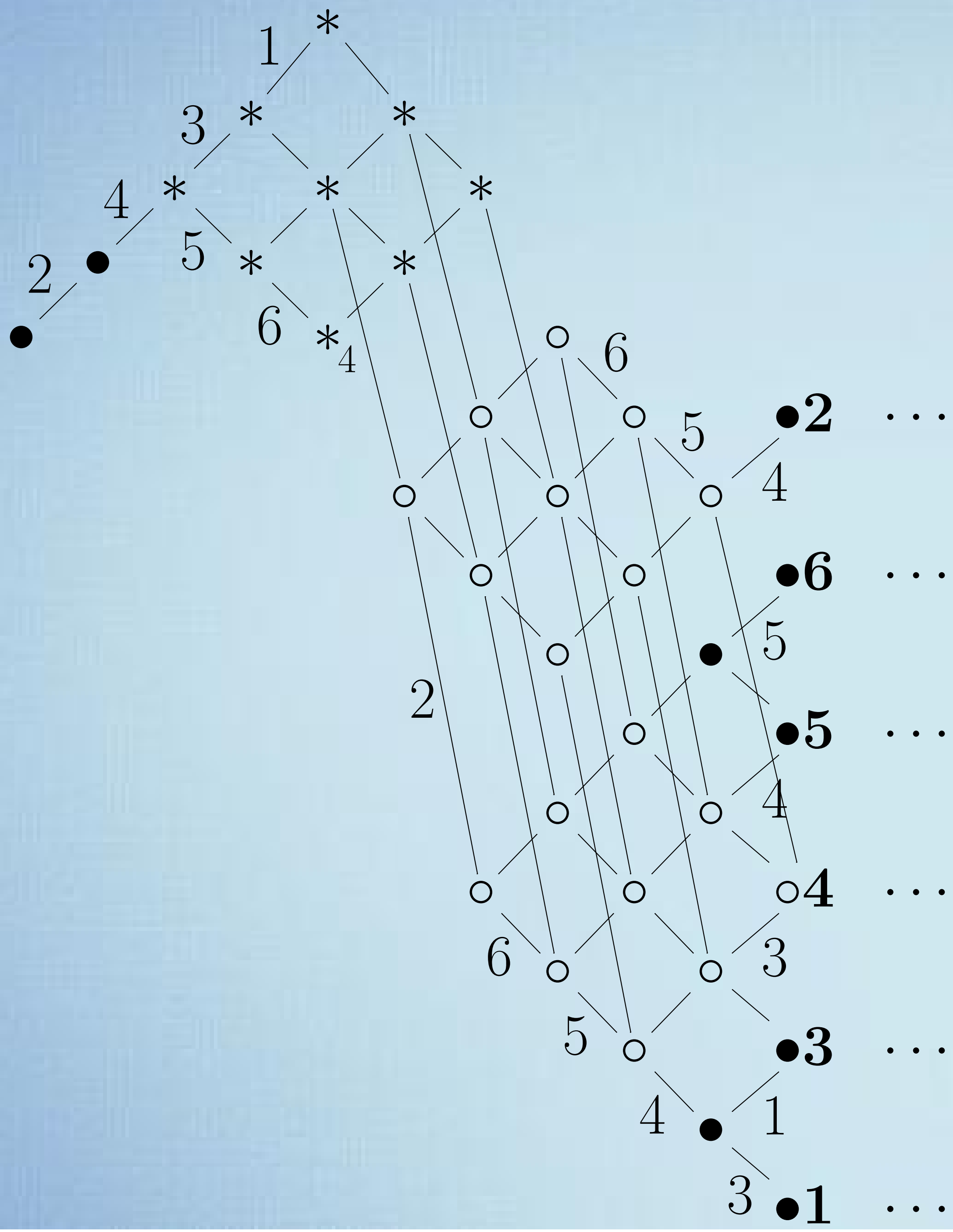
$\Delta = 3A_2$



$\mathbf{E_6}$, adjoint



orbits of $\mathbf{W(3A_2)}$



Summary for $3A_2 \le E_6$

Orbits $W(3A_2)$ splits $E_6 \setminus 3A_2$ into:

$\mathbf{O_1}$: 9 positive and 18 negative roots, represented by $\begin{smallmatrix} *&*&2&*&* \\ &&*&& \end{smallmatrix}$, $-\begin{smallmatrix} *&*&1&*&* \\ &&*&& \end{smallmatrix}$

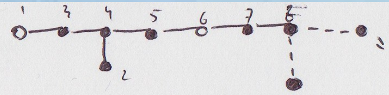
$\mathbf{O_2}$: 18 positive and 9 negative roots, represented by $\begin{smallmatrix} *&*&1&*&* \\ &&*&& \end{smallmatrix}$, $-\begin{smallmatrix} *&*&2&*&* \\ &&*&& \end{smallmatrix}$

Relations for ideals: $I_1^2 \le I_2, I_2^2 \le I_1$

Chevalley commutator formula for simply laced root system:

$$[x_\alpha(\xi), x_\beta(\zeta)] = x_{\alpha+\beta}(\xi\zeta)$$

$2D_4 \le E_8$



$= -\max(E_8) =: m_1 = -\,{}^{246\,5432}_{\quad 3}$

$-\max(D_8) = m_1 + \alpha_3 + \alpha_2 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + 2\alpha_8 =$

$= -\left({}^{246\,5432}_{\quad 3} - {}^{0122222}_{\quad 1}\right) =$

$= -\,{}^{2343210}_{\quad 2} =: m_2$

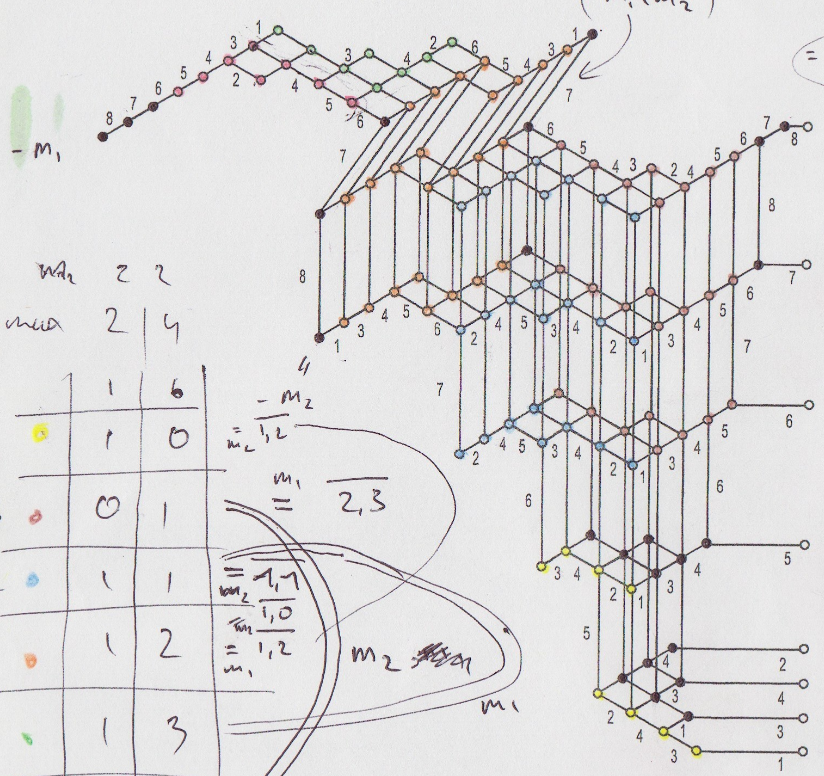


FIGURE 24. (E_8, ϖ_8)

	m_2	m_2
m_1	2	4

		1	6
A	(yellow)	1	0
B	(brown)	0	1
C	(blue)	1	1
	(orange)	1	2
	(green)	1	3
	(red)	2	3

3 орбиты

$AB \le C$
$AC \le B$
$BC \le A$

${}^{2343210}_{\quad 2} - m_1$

$m_1 : (6,3)$ • 8
$m_2 : (3,0)$ •

цвет	4	7
A	1	0
B	2	0
C	0	1
D	1	1
E	2	1
G	3	1
D	4	1
F	2	2
G	3	2
И	4	2
F	5	2
G	6	2

4	7	
−1	0	B
−2	0	A
0	−1	G
−1	−1	F
−2	−1	И
−3	−1	G
−4	−1	F
−2	−2	D
−3	−2	C
−4	−2	E
−5	−2	D
−6	−2	C

8 орбит

$A_2 \le E_8$

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$-\max(E_6) = m_2 = {}^{123\,2100}_{\quad 2}$

$-\max(E_8) = {}^{246\,5432}_{\quad 3} = m_1$

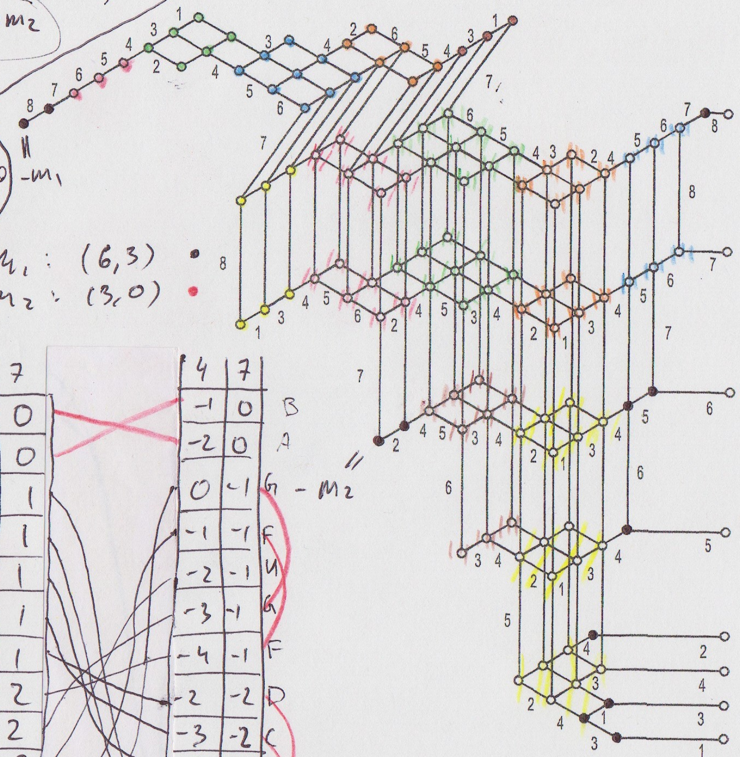

FIGURE 24. (E_8, ϖ_8)

$A^2 \le B$, $AC \le D$, $BC \le E$, $D^2 \le F$, $AD \le E$, $BD \le C$, $E^2 \le И$, $AE \le C$, $BE \le D$, $CE \le F$, $DE \le G$, $C^2 \le G$, $CD \le И$, $AF \le G$, $BF \le И$, $AG \le И$, $BG \le F$, $AИ \le F$, $BИ \le G$; $B^2 \le A$, $F^2 \le D$, $И^2 \le E$, $GИ \le D$, $FИ \le C$, $G^2 \le C$, $GF \le E$;

Brief results summary for root systems E_6 and E_7

Δ	#	Δ	#	Δ	#
$3A_3 \leq E_6$	2	$D_5 \leq E_6$	2	$A_5 + A_1 \leq E_6$	1
$D_4 \leq E_6$	6	$2A_2 + A_1 \leq E_6$	6	$A_3 + 2A_1 \leq E_6$	5
$A_7 \leq E_7$	1	$A_5 + A_2 \leq E_7$	2	$E_6 \leq E_7$	2
$A_6 \leq E_7$	4	$A_4 + A_2 \leq E_7$	6	$3A_2 \leq E_7$	8
$D_6 + A_1 \leq E_7$	1	$D_5 + A_1 \leq E_7$	4	$A_3 + A_2 + A_1 \leq E_7$	8
$(A_1 + A_5)' \leq E_7$	5	$(A_1 + A_5)'' \leq E_7$	6	$D_4 + 3A_1 \leq E_7$	3
$A_4 + A_1 \leq E_7$	12				

Remaining steps

1. Description of normalizer
2. Exctraction of unipotents
3. Inclusion in the normalizer

Main result for a simple case

Theorem: Let R be a commutative ring and H be a subgroup in $G(E_6, R)$ containing $E(A_5 + A_1, R)$, then there exists a unique ideal I such that

$$E(A_5 + A_1, R)E(E_6, I) \leq H \leq N_{G(E_6,R)}(E(A_5 + A_1, R)E(E_6, I))$$

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Thank You!