Overgroups of Subsystem Subgroups in Exceptional Groups
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General notations

Φ – irreducible reduced root system of type $E_6, E_7, E_8, G_2, F_4$.

$\Delta$ – subsystem of $\Phi$ (not necessary irreducible)

$R$ – commutative associative ring with unity

$G(\Phi, R)$ – Chevalley group of type $\Phi$

$E(\Delta, R)$ – Elementary Chevalley group of type $\Delta$

$W(\Delta)$ – Weyl group of root system $\Delta$ considered as a subgroup of $W(\Phi)$
Main conjecture

Let $\Phi$ be an irreducible reduced root system, $\Delta \leq \Phi$, satisfying condition (*), $R$ - commutative ring and $H$ be a subgroup of $G(\Phi, R)$ such that

$$E(\Delta, R) \leq H \leq G(\Phi, R),$$

then there exist a unique $k$-tuple of ideals $(I_\omega)_{\omega=1}^k$ in $R$ such that

$$E^*(\Delta, R, I_1, \ldots, I_k) \leq H \leq NG(E^*(\Delta, R, I_1, \ldots, I_k))$$

$k$ – number of orbits of $W(\Delta)$ on $\Phi \setminus \Delta$

$E^*$ – some subgroup in $G(\Phi, R)$, defined by ideals $I_1, \ldots, I_k$ and containing $E(\Delta, R)$
Description of ideals

Let $E(\Delta, R) \leq H \leq G(\Phi, R)$ For any root $\alpha \in \Phi$
define $I_\alpha := \{\xi \in R|x_\alpha(\xi) \in H\}$

**Proposition**

1. for any $\alpha \in \Phi$ \hspace{1em} $I_\alpha$ is an ideal in $R$
2. for any $\alpha, \beta$ lying in the same orbit of action of $W(\Delta)$ on $\Phi \setminus \Delta$ \hspace{1em} $I_\beta = I_\alpha = I_{[\alpha]}$

**Idea of the proof:** use Chvalley comutator formula:

$$[x_\alpha(\xi), x_\beta(\zeta), x_{-\beta}(1)] = x_\alpha(\xi\zeta)$$

(true for any roots $\alpha, \beta$, such that $\alpha + \beta$ is a root itself)
Description of subgroup $E^*$, condition (*)

$k = 1$:

$E^*(\Delta, R, I) = E(\Delta, R)E(\Phi, I)$

$k > 1$:

$E^*(\Delta, R, I_1, \ldots, I_k) = \langle x_\alpha(\xi) | \alpha \in \Phi, \xi \in I_{[\alpha]} \rangle$ - generalization of the notion of the elementary net subgroup.

**Remark:** difficulty of computations depends on the number of orbits

Condition (*):

weak: $\Delta^\perp = \emptyset$

strong: for each $\alpha$ in $\Phi \setminus \Delta$ there exists $\beta$ in an irreducible component of $\Delta$ of rank at least 2 such that $\alpha + \beta$ is a root in $\Phi$. 
Example:
\[ \Phi = E_6 \]
\[ \Delta = 3A_2 \]
E₆, adjoint
orbits of $W(3A_2)$
Summary for $3A_2 \leq E_6$

Orbits $W(3A_2)$ splits $E_6 \setminus 3A_2$ into:

$O_1$: 9 positive and 18 negative roots, represented by $**2**$, $-**1**$

$O_2$: 18 positive and 9 negative roots, represented by $**1**$, $-**2**$

Relations for ideals: $I_1^2 \leq I_2$, $I_2^2 \leq I_1$

Chevalley commutator formula for simply laced root system:

$$[x_\alpha(\xi), x_\beta(\zeta)] = x_{\alpha + \beta}(\xi\zeta)$$
**Brief results summary for root systems $E_6$ and $E_7$**

<table>
<thead>
<tr>
<th>$\Delta$</th>
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<td>2</td>
<td>$D_5 \leq E_6$</td>
<td>2</td>
<td>$A_5 + A_1 \leq E_6$</td>
<td>1</td>
</tr>
<tr>
<td>$D_4 \leq E_6$</td>
<td>6</td>
<td>$2A_2 + A_1 \leq E_6$</td>
<td>6</td>
<td>$A_3 + 2A_1 \leq E_6$</td>
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<tr>
<td>$A_7 \leq E_7$</td>
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<td>$A_5 + A_2 \leq E_7$</td>
<td>2</td>
<td>$E_6 \leq E_7$</td>
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<tr>
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<td>$A_4 + A_2 \leq E_7$</td>
<td>6</td>
<td>$3A_2 \leq E_7$</td>
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<tr>
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<td>1</td>
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<td>$A_3 + A_2 + A_1 \leq E_7$</td>
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<tr>
<td>$(A_1 + A_5)’ \leq E_7$</td>
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<td>$(A_1 + A_5)'' \leq E_7$</td>
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<td>$D_4 + 3A_1 \leq E_7$</td>
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<tr>
<td>$A_4 + A_1 \leq E_7$</td>
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Remaining steps

1. Description of normalizer
2. Extraction of unipotents
3. Inclusion in the normalizer
Theorem: Let $R$ be a commutative ring and $H$ be a subgroup in $G(E_6, R)$ containing $E(A_5 + A_1, R)$, then there exists a unique ideal $I$ such that

$$E(A_5 + A_1, R)E(E_6, I) \leq H \leq N_{G(E_6, R)}(E(A_5 + A_1, R)E(E_6, I))$$
Bibliography


[PSV] E. Plotkin, A. Semenov, and N. Vavilov. Visual basic representa-


Thank You!