# On groups of odd order admitting an elementary 2-group of automorphisms

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# Preliminaries

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$A$ is fixed-point-free if $C_G(A) = 1$.

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**Shumyatsky (1988)**

If $C_G(A) = 1$, then $G$ has a normal series

$$G = N_0 \geq N_1 \geq \cdots \geq N_n = 1$$

where $N_{i-1}/N_i$ is nilpotent of class bounded in terms of $k$ and $i$.

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### Shumyatsky, Sica (2010)

There exists a number $s = s(k, n)$ depending only on $k$ and $n$ with the following property.
Let $R$ be a normal $A$-invariant subgroup of $G$ such that $C_R(A) = 1$.
Set $N = \cap_{a \in A^\#}[G, a]$. Then $[R, \underbrace{N, \ldots, N}_{s}] = 1$.

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### Corollary

If $C_G(A) = 1$. Then $N = \cap_{a \in A^\#}[G, a]$ is nilpotent of $\{k, n\}$-bounded class.

# Centralizer of exponent $e$

Let $G$ be a finite group of odd order and derived length $k$.
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## Question

What we can say when $\exp(C_G(A)) = e$?

# Centralizer of exponent $e$

**Theorem 1**

Let $G$ be a finite group of odd order and of derived length $k$. Suppose that $G$ admits an elementary abelian group $A$ of automorphisms of order $2^n$ such that $C_G(A)$ has exponent $e$. Then $G$ has a normal series

$$G = G_0 \geq T_0 \geq G_1 \geq T_1 \geq \cdots \geq G_n \geq T_n = 1$$

such that the quotients $G_i/T_i$ have $\{k, e, n\}$-bounded exponent and the quotients $T_i/G_{i+1}$ are nilpotent of $\{k, e, n\}$-bounded class.

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**Lemma 2**

Let $G$ be a finite group of odd order and of derived length $k$. Suppose that $G$ admits an automorphism $a$ of order 2 such that $C_G(a)$ has exponent $e$. Then $[G, a]'$ has $\{e, k\}$-bounded exponent.

Let $G$ be a finite group of odd order and derived length $k$. Suppose that $A$ is a subgroup of automorphisms of $G$, $A$ elementary abelian of order $2^n$.

## Proposition 1

If $C_G(A)$ has exponent $e$, then $N = \bigcap_{a \in A^\#} [G, a]$ is an extension of a group of $\{k, e, n\}$-bounded exponent by a nilpotent group of $\{k, e, n\}$-bounded class.

## Sketch of the proof of Theorem 1:

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is the required series.
Then we argue by induction on $n$.
Since $a$ acts trivially on $G/[G, a]$, for any $a \in A \setminus \{1\}$, $A$ induces on $G/[G, a]$ a group of automorphisms $B$ of order $2^{n-1}$ with $C_{G/[G,a]}$ of exponent $e$.

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Put $K = \prod_{a \in A^{\#}} G/[G, a]$. We can extend the action of $B$ on $K$ in natural way and $C_K(B)$ has exponent $e$.

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Put $N = \bigcap_{a \in A^{\#}} [G,a]$, since $G/N$ embeds in $K$, by induction $G/N$ has the required series.
By Proposition 1 the subgroup $N$ is an extension of a group of $\{k,e,n\}$-bounded exponent by a nilpotent group of $\{k,e,n\}$-bounded class, and the proof is complete.

# Centralizers are finite exponent-by-nilpotent

**Khukhro, Shumyatsky (1999)**

Let $q$ a prime number and let $G$ be a finite group of order coprime with $q$. Suppose that $G$ admits an elementary abelian group $A$ of automorphisms of order $q^2$ such that $C_G(a)$ has exponent dividing $e$ for any $a \in A^{\#}$. Then $G$ has $\{e, q\}$-bounded exponent.

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Let $G$ be a finite group of odd order and derived length $k$.
Suppose that $V$ is a subgroup of automorphisms of $G$,
$V$ elementary abelian of order 4.

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Let $G$ be a finite group of odd order and derived length $k$.
Suppose that $V$ is a subgroup of automorphisms of $G$,
$V$ elementary abelian of order 4.

## Question

What we can say when $\gamma_c(C_G(v))$ has exponent dividing $e$ for all automorphisms $v \in V^{\#}$?

# Centralizers are finite exponent-by-nilpotent

Let $G$ be a finite group of odd order and derived length $k$. Suppose that $V$ is a subgroup of automorphisms of $G$, $V$ elementary abelian of order 4.

## Proposition 2

If $\gamma_c(C_G(v))$ has exponent dividing $e$ for all $v \in V^\#$. Then $N = \cap_{v \in V^\#}[G, v]$ is an extension of a group of $\{e, c, k\}$-bounded exponent by a nilpotent group of $\{e, c, k\}$-bounded class.

# Centralizers are finite exponent-by-nilpotent

**Theorem 2**

Let $G$ be a finite group of odd order and of derived length $k$. Suppose that a four-group $V$ acts on $G$ in such way that $\gamma_c(C_G(v))$ has exponent dividing $e$ for all $v \in V^{\#}$. Then $G$ has a normal series

$$G = T_4 \geq T_3 \geq T_2 \geq T_1 \geq T_0 = 1$$

such that the quotients $T_4/T_3$ and $T_2/T_1$ are nilpotent of $\{e, c, k\}$-bounded class and the quotients $T_3/T_2$ and $T_1$ have $\{e, c, k\}$-bounded exponent.

# Thanks!