Unitriangular factorisations of Chevalley groups

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Joint work with Nikolai Vavilov and B. Sury

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Problem:

For given Chevalley group $G = G(\Phi, R)$ find the shortest representation of the form

$$G = UU^{-}\ldots U^{\pm},$$

where $U^{\pm} = U^{\pm}(\Phi, R)$ are the unipotent radicals of standard Borel subgroup B or its opposite B^{-}.

Chevalley group $G(\Phi, R)$ admits unitriangular factorisation if an only if

1. $G(\Phi, R) = E(\Phi, R)$,
2. $E(\Phi, R)$ has bounded width with respect to root elements.

Why?

- Over a finite field K of $char\, K = p$ the subgroup $U(\Phi, K)$ is a Sylow p-subgroup.
- Any element of $U(n, R)$, $n \geqslant 3$ is the product of 2 commutators in $E(n, R)$.

Known results:

- $SL(n,R) = UU^-UU^-$ over rings of stable rank 1 [Hyman Bass, 1964].

- $G = \underbrace{UU^-\dots U^-}_{8}$ over semi-local rings for all Chevalley groups [Andrei Rapinchuk, Igor Rapinchuk, 2010].

- $G = UU^-UU^-U$ over finite fields for all Chevalley group of normal and twisted types, including Suzuki and small Ree groups [Laslo Babai, Nikolai Nikolov, Laslo Pyber, 2008].

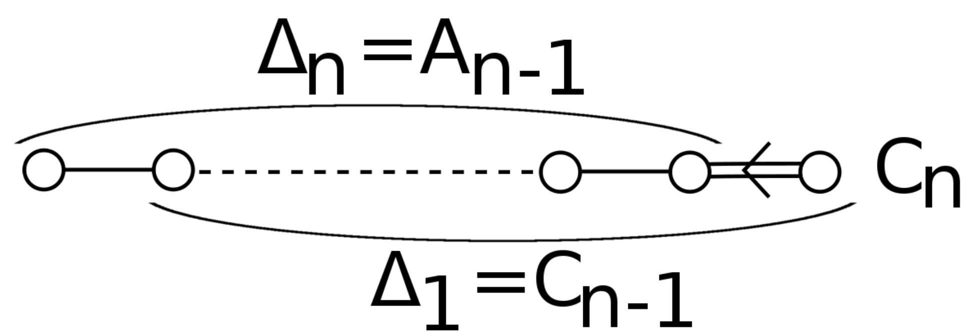
New result:

$$G(\Phi, R) = UU^-UU^-$$

over rings of stable rank 1 for Chevalley groups of normal and twisted types, except for groups of type ${}^2A_{2n}$.

Definition (terminal root subsystems):

Let Φ be a root system and Π a set of fundamental roots. Then a root system, spanned by all fundamental roots except one of the terminals, is called a terminal subsystem.



Rank reduction theorem (after Oleg Tavgen):

If unitriangular factorisation of length N holds for Chevalley groups, corresponding to terminal subsystems of root system ${}^{\sigma}\Phi$, then it holds for group $G({}^{\sigma}\Phi, R)$ and has the same length N.

Factorisation for $SL(2, R)$ follows immediately from the condition $sr(R) = 1$.

Thus the following groups have factorisation of length 4:

- all Chevalley groups of normal type
- twisted groups of types ${}^2A_{2n+1}$, ${}^2D_n$, ${}^3D_4$, ${}^2E_6$.

Groups of type ${}^2A_{2n}$:

$$G\left({}^2A_{2n}, R\right) = UU^-UU^-$$

- for $R = \mathbb{F}_{q^2}$ a finite involutary field
- for $R = \mathbb{C}$ with complex conjugation as an involution.

Theorem (Sury):

Let $p \in \mathbb{Z}$ be a prime. Then under assumption of Generalised Riemann's Hypothesis simply connected Chevalley group $G\left(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\right)$ admits unitriangular factorisation

$$G\left(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\right) = \left(U\left(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\right) \cdot U^{-}\left(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\right)\right)^{3}$$

of length 6.

More details can be found in

N. A. Vavilov, A. V. Smolensky, B. Sury, *Unitriangular factorisations of Chevalley groups*, `arXiv:1107.5414v1 [math.GR]`

Thank you!