Unitriangular factorisations of Chevalley groups

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Problem:

For given Chevalley group $G = G(\Phi, R)$ find the shortest representation of the form

$$G = UU^- \ldots U^\pm,$$

where $U^\pm = U^\pm(\Phi, R)$ are the unipotent radicals of standard Borel subgroup $B$ or its opposite $B^-$. 
Chevalley group $G(\Phi, R)$ admits unitriangular factorisation if and only if

1. $G(\Phi, R) = E(\Phi, R)$,

2. $E(\Phi, R)$ has bounded width with respect to root elements.
Why?

• Over a finite field $K$ of $\text{char } K = p$ the subgroup $U(\Phi, K)$ is a Sylow $p$-subgroup.

• Any element of $U(n, R)$, $n \geq 3$ is the product of 2 commutators in $E(n, R)$. 
Known results:

- $SL(n, R) = UU^-UU^-$ over rings of stable rank 1 [Hyman Bass, 1964].

- $G = \underbrace{UU^-}_{8}$ over semi-local rings for all Chevalley groups [Andrei Rapinchuk, Igor Rapinchuk, 2010].

- $G = UU^-UU^-U$ over finite fields for all Chevalley group of normal and twisted types, including Suzuki and small Ree groups [Laslo Babai, Nikolai Nikolov, Laslo Pyber, 2008].
New result:

$$G(\Phi, R) = UU^-UU^-$$

over rings of stable rank 1 for Chevalley groups of normal and twisted types, except for groups of type $^2A_{2n}$. 
Definition (terminal root subsystems):

Let Φ be a root system and Π a set of fundamental roots. Then a root system, spanned by all fundamental roots except one of the terminals, is called a terminal subsystem.
Rank reduction theorem (after Oleg Tavgen):

If unitriangular factorisation of length $N$ holds for Chevalley groups, corresponding to terminal subsystems of root system $\sigma \Phi$, then it holds for group $G(\sigma \Phi, R)$ and has the same length $N$. 
Factorisation for $SL(2, R)$ follows immediately from the condition $sr(R) = 1$.

Thus the following groups have factorisation of length 4:

- all Chevalley groups of normal type
- twisted groups of types $^2A_{2n+1}$, $^2D_n$, $^3D_4$, $^2E_6$. 
Groups of type $^2A_{2n}$:

$$G\left(^2A_{2n}, R\right) = UU^-UU^-$$

- for $R = \mathbb{F}_{q^2}$ a finite involutary field
- for $R = \mathbb{C}$ with complex conjugation as an involution.
Theorem (Sury):

Let $p \in \mathbb{Z}$ be a prime. Then under assumption of Generalised Riemann’s Hypothesis simply connected Chevalley group $G\bigl(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\bigr)$ admits unitriangular factorisation

$$G\bigl(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]\bigr) = \left(U(\Phi, \mathbb{Z}\left[\frac{1}{p}\right]) \cdot U^{-}(\Phi, \mathbb{Z}\left[\frac{1}{p}\right])\right)^3$$

of length 6.
More details can be found in


Thank you!