The Coxeter complex and the Euler characteristics of a Hecke algebra

Tommaso Terragni

A joint work with Thomas Weigel

Università di Milano–Bicocca

Ischia Group Theory 2012

Coxeter groups

A **Coxeter system** (W, S) consists of a group W generated by a finite set S of involutions subject to relations of the form $(st)^{m_{s,t}}$, for $s \neq t \in S$ and suitable $m_{s,t} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$.

For $I \subseteq S$ the **parabolic subsystem** is (W_I, I), where $W_I = \langle I \rangle$.

Let $\ell\colon W \longrightarrow \mathbb{N}_0$ be the **length function** w.r.t. the generating system S.

Define the **Poincaré series** of (W, S) as

$$p_{(W,S)}(t) = \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[\![t]\!].$$

It is well-known that $p_{(W,S)}(t)$ is a rational function and, for $|W| = \infty$,

$$\frac{1}{p_{(W,S)}(t)} = \sum_{I \subsetneq S} (-1)^{|S \setminus I| - 1} \frac{1}{p_{(W_I, I)}(t)}. \qquad (\star)$$

Euler characteristic of Coxeter groups

Theorem (Serre 71)

If (W, S) is a Coxeter group then W is of type WFL (hence VFP): thus, it admits an Euler characteristic χ_W. Moreover,

$$\chi_W = \frac{1}{p_{(W,S)}(1)}.$$

Euler characteristic of Coxeter groups

Theorem (Serre 71)

If (W, S) is a Coxeter group then W is of type WFL (hence VFP): thus, it admits an Euler characteristic χ_W. Moreover,

$$\chi_W = \frac{1}{p_{(W,S)}(1)}.$$

A recursive, alternating-sum formula (like $(\star)$) is often associated to the **Euler characteristic** of a suitably defined **algebraic object**. Thus, the following question is natural:

Main problem

- Does there exist an **object** (associated with (W, S)) such that its Euler characteristics coincides with the inverse of the Poincaré series $p_{(W,S)}(t)$?
- What is a sensible **definition of Euler characteristic** for such objects?

Hecke algebras

Datum: a Coxeter system (W, S), a commutative ring R and a parameter $q \in R$.

R-Hecke algebra $\mathcal{H}_q(W, S)$

Let $\mathcal{H}$ be the free R-module with basis $\{ T_w \mid w \in W \}$ and associative R-linear multiplication defined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$ and $w \in W$. The function ℓ is the length function of (W, S).

Hecke algebras

Datum: a Coxeter system (W, S), a commutative ring R and a parameter $q \in R$.

R-Hecke algebra $\mathcal{H}_q(W, S)$

Let $\mathcal{H}$ be the free R-module with basis $\{ T_w \mid w \in W \}$ and associative R-linear multiplication defined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ q T_{sw} + (q-1) T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$ and $w \in W$. The function ℓ is the length function of (W, S).

For $q = 1 \in R$, one has $\mathcal{H}_1(W, S) \simeq R[W]$. Thus, a Hecke algebra can be considered as a **"generic form"** or a **"deformation"** of the group algebra of the Coxeter group.

Hecke algebras

Datum: a Coxeter system (W, S), a commutative ring R and a parameter $q \in R$.

R-Hecke algebra $\mathcal{H}_q(W, S)$

Let $\mathcal{H}$ be the free R-module with basis $\{ T_w \mid w \in W \}$ and associative R-linear multiplication defined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$ and $w \in W$. The function ℓ is the length function of (W, S).

For $q = 1 \in R$, one has $\mathcal{H}_1(W, S) \simeq R[W]$. Thus, a Hecke algebra can be considered as a **"generic form"** or a **"deformation"** of the group algebra of the Coxeter group. Hecke algebras (and their parabolic subalgebras) are canonically equipped with

- A free R-**basis** $\{T_w \mid w \in W\}$,
- a linear character $\varepsilon_q \colon \mathcal{H} \longrightarrow R$, $\varepsilon_q(T_w) = q^{\ell(w)}$ (**trivial module** R_q),
- an **antipodal map** $\_^\natural \colon \mathcal{H}^{\mathrm{op}} \longrightarrow \mathcal{H}$, $(T_w)^\natural = T_{w^{-1}}$,
- a poset of parabolic subalgebras $\mathcal{H}_I$.

Euler algebras, 1

We propose the definition of **Euler algebras** to single out the sufficient conditions for an (associative) algebra to **admit a notion of Euler characteristic**.
This mimicks the behaviour of group algebras.

Euler algebras, 1

We propose the definition of **Euler algebras** to single out the sufficient conditions for an (associative) algebra to **admit a notion of Euler characteristic**.
This mimicks the behaviour of group algebras.

Definition

Euler R-algebra Let $\mathbf{A}$ be an associative R-algebra, together with

- be a free R-basis $\mathcal{B} \ni 1$,
- an antipodal map $\_^{\natural} \colon \mathcal{H}^{\mathrm{op}} \longrightarrow \mathcal{H}$ such that $\mathcal{B}^{\natural} = \mathcal{B}$,
- an augmentation $\varepsilon \colon \mathbf{A} \longrightarrow R$ onto a one-dimensional left $\mathbf{A}$-module R_{ε}, such that $\varepsilon \circ \_^{\natural} = \varepsilon$,
- a canonical trace function $\mu \colon \mathbf{A}/[\mathbf{A}, \mathbf{A}] \longrightarrow R$ defined by $\mu(a) = \delta_{1,a}\varepsilon(a)$ for $a \in \mathcal{B}$.

Suppose further that R_q is a left $\mathbf{A}$-module of type FP, i.e., it admits a finite, projective $\mathbf{A}$-resolution.
Then, the 5-tuple $\mathbf{A} = (\mathbf{A}, \mathcal{B}, \_^{\natural}, \varepsilon, \mu)$ is an **Euler algebra**.

Euler algebras, 2

Let M be a left $\mathbf{A}$-module of type FP (with a finite, projective resolution $P_\bullet$). The **Hattori–Stallings rank** of M is defined as the element

$$r_M = \sum_{k\geq 0} (-1)^k p_k^*(p_k) + [\mathbf{A},\mathbf{A}] \in \mathbf{A}/[\mathbf{A},\mathbf{A}],$$

where $\gamma(p_k^* \otimes p_k) = \mathrm{id}_{P_k}$ for the natural isomorphism $\gamma\colon P_k^* \otimes P_k \longrightarrow \mathrm{End}_{\mathsf{A}}(P_k)$. This only depends on the module M, and not on $P_\bullet$.
If $\mathbf{B} \leq \mathbf{A}$ is an inclusion of Euler algebras and if $\mathbf{A}$ is $\mathbf{B}$-flat, then, for $\mathbf{B}$-module M

$$r_{\mathrm{ind}_{\mathsf{B}}^{\mathsf{A}} M} = r_M + [\mathbf{A},\mathbf{A}].$$

Euler algebras, 2

Let M be a left $\mathbf{A}$-module of type FP (with a finite, projective resolution $P_\bullet$). The **Hattori–Stallings rank** of M is defined as the element

$$r_M = \sum_{k \geq 0} (-1)^k p_k^*(p_k) + [\mathbf{A}, \mathbf{A}] \in \mathbf{A}/[\mathbf{A}, \mathbf{A}],$$

where $\gamma(p_k^* \otimes p_k) = \mathrm{id}_{P_k}$ for the natural isomorphism $\gamma\colon P_k^* \otimes P_k \longrightarrow \mathrm{End}_{\mathrm{A}}(P_k)$. This only depends on the module M, and not on $P_\bullet$.
If $\mathbf{B} \leq \mathbf{A}$ is an inclusion of Euler algebras and if $\mathbf{A}$ is $\mathbf{B}$-flat, then, for $\mathbf{B}$-module M

$$r_{\mathrm{ind}_{\mathrm{B}}^{\mathrm{A}} M} = r_M + [\mathbf{A}, \mathbf{A}].$$

A **trace function** is a map of R-modules $\mu\colon \mathbf{A}/[\mathbf{A}, \mathbf{A}] \longrightarrow R$.

Definition (Euler characteristic)

For an R-Euler algebra $\mathbf{A} = (\mathbf{A}, \mathcal{B}, \_^{\natural}, \varepsilon, \mu)$, define

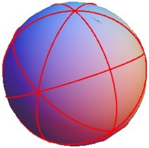
$$\chi_{\mathrm{A}} = \mu(r_{R_\varepsilon}).$$

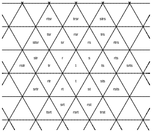
Hecke algebras are Euler, 1

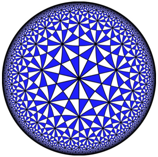
We already know that Hecke algebras admit a free basis $\{T_w \mid w \in W\}$, an antipode $T_w \mapsto T_{w^{-1}}$, an augmentation ε_q and a canonical trace function μ defined by $\mu(T_w) = \delta_{1,w}$.

Last step: proving that the trivial module R_q is of type FP.

Coxeter groups have a very rich geometry, and from the Coxeter complex one can produce resolutions of the trivial module.







Hecke algebras are Euler, 1

We already know that Hecke algebras admit a free basis $\{ T_w \mid w \in W \}$, an antipode $T_w \mapsto T_{w^{-1}}$, an augmentation ε_q and a canonical trace function μ defined by $\mu(T_w) = \delta_{1,w}$.

Last step: proving that the trivial module R_q is of type FP.

Coxeter groups have a very rich geometry, and from the Coxeter complex one can produce resolutions of the trivial module. Unfortunately, **there is no obvious geometry** for Hecke algebras,

Hecke algebras are Euler, 1

We already know that Hecke algebras admit a free basis $\{T_w \mid w \in W\}$, an antipode $T_w \mapsto T_{w^{-1}}$, an augmentation ε_q and a canonical trace function μ defined by $\mu(T_w) = \delta_{1,w}$.

Last step: proving that the trivial module R_q is of type FP.

Coxeter groups have a very rich geometry, and from the Coxeter complex one can produce resolutions of the trivial module. Unfortunately, **there is no obvious geometry** for Hecke algebras, but we constructed a "module-theoretic" analogue $C_\bullet$ of the Coxeter complex Σ (canonical up to the choice of a total order $<$ on S).

Let $I \subseteq S$ and denote $\mathbf{ind}_I^S$ the induction from left $\mathcal{H}_I$- to left $\mathcal{H}$-modules. If W_I is finite and $p_{(W,S)}(q) \in R$ is invertible, then let $e_I = \frac{1}{p_{(W,S)}(q)} \sum_{w \in W_I} T_w$. Then

$$e_I^2 = e_I \quad \text{and} \quad \mathbf{ind}_I^S R_q \simeq \mathcal{H} e_I.$$

Moreover the Hattori–Stallings rank of $\mathbf{ind}_I^S R_q$ is $e_I + [\mathcal{H}_I, \mathcal{H}_I]$.

Hecke algebras are Euler, 2

For $I \subseteq S$ and $s \in S$, define $\deg(I) = |S| - |I| - 1$ and the sign map $\operatorname{sgn}(s, I) = (-1)^{|\{ t \in S \setminus I \mid t < s \}|}$.

The Hecke–Coxeter complex $(C_\bullet, \partial_\bullet)$.

$$(C_\bullet, \partial_\bullet) = 0 \longrightarrow C_{|S|-1} \xrightarrow{\partial_{|S|-1}} C_{|S|-2} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where $C_k = \coprod_{\deg(I)=k} \mathbf{ind}_I^S R_q$ and

$$\partial_k(T_w \eta_I) = \sum_{\substack{s \in S \setminus I \\ J = I \sqcup \{s\}}} \operatorname{sgn}(s, I) \varepsilon_q(T_{w_J}) T_{w^J} \eta_J.$$

Hecke algebras are Euler, 2

For $I \subseteq S$ and $s \in S$, define $\deg(I) = |S| - |I| - 1$ and the sign map $\operatorname{sgn}(s, I) = (-1)^{|\{t \in S \setminus I \mid t < s\}|}$.

The Hecke–Coxeter complex $(C_\bullet, \partial_\bullet)$.

$$(C_\bullet, \partial_\bullet) = 0 \longrightarrow C_{|S|-1} \xrightarrow{\partial_{|S|-1}} C_{|S|-2} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

where $C_k = \coprod_{\deg(I)=k} \mathbf{ind}_I^S R_q$ and

$$\partial_k(T_w \eta_I) = \sum_{\substack{s \in S \setminus I \\ J = I \sqcup \{s\}}} \operatorname{sgn}(s, I) \varepsilon_q(T_{w_J}) T_{w^J} \eta_J.$$

Theorem A

Let (W, S) be a Coxeter group with $2 \leq |S| < \infty$ and let C be the Hecke–Coxeter complex of the R-Hecke algebra $\mathcal{H} = \mathcal{H}_q(W, S)$.

- *If (W, S) is spherical, then $H_k(C) = 0$ unless $k = 0$ or $k = |S| - 1$. Moreover, $H_0(C) \simeq R_q$ and $H_{|S|-1}(C) \simeq R_{-1}$.*
- *If (W, S) is non-spherical then C is acyclic with $H_0(C) \simeq R_q$.*

Euler characteristic of Hecke algebras

Suppose

$$p_{(W_I,I)}(q) \text{ is invertible for all } I \text{ such that } W_I \text{ is finite.} \qquad (**)$$

Then, by a standard argument, the trivial module R_q is of type FP.

Euler characteristic of Hecke algebras

Suppose

$$p_{(W_I,I)}(q) \text{ is invertible for all } I \text{ such that } W_I \text{ is finite.} \qquad (**)$$

Then, by a standard argument, the trivial module R_q is of type FP.

Proposition B

Any Hecke algebra $\mathcal{H}_q(W,S)$ is of type FP if the condition $(**)$ on q holds true. With the canonical structure $\mathcal{H} = (\mathcal{H}, \_^{\natural}, \varepsilon_q, \mathcal{B}, \mu)$ described before, **any Hecke algebra satisfying** $(**)$ **is an Euler algebra**.

When $R = \mathbb{Z}[\![q]\!]$ and $|W| = \infty$ one thus has

$$\chi_{\mathcal{H}} = \sum_{I \subsetneq S} (-1)^{|S \setminus I|-1} \frac{1}{p_{(W_I,I)}(q)} = \frac{1}{p_{(W,S)}(q)}.$$

Euler characteristic of Hecke algebras

Suppose

$$p_{(W_I,I)}(q) \text{ is invertible for all } I \text{ such that } W_I \text{ is finite.} \qquad (**)$$

Then, by a standard argument, the trivial module R_q is of type FP.

Proposition B

Any Hecke algebra $\mathcal{H}_q(W,S)$ is of type FP if the condition $(**)$ on q holds true. With the canonical structure $\mathcal{H} = (\mathcal{H}, \_^{\natural}, \varepsilon_q, \mathcal{B}, \mu)$ described before, **any Hecke algebra satisfying $(**)$ is an Euler algebra**.

When $R = \mathbb{Z}[\![q]\!]$ and $|W| = \infty$ one thus has

$$\chi_{\mathcal{H}} = \sum_{I \subsetneq S} (-1)^{|S \setminus I|-1} \frac{1}{p_{(W_I,I)}(q)} = \frac{1}{p_{(W,S)}(q)}.$$

Dealing separately also the (elementary) case $|W| < \infty$ one has

Corollary

If $R = \mathbb{Z}[\![q]\!]$ and $\mathcal{H}$ is the R-Hecke algebra with Coxeter system (W,S), then

$$\chi_{\mathcal{H}}\, p_{(W,S)}(q) = 1$$

References

- H. Bass, *Euler characteristics and characters of discrete groups*
- N. Bourbaki, *Groupes et algèbres de Lie, Ch. 4 à 6.*
- K.S. Brown, *Cohomology of groups.*
- M. Geck & G. Pfeiffer, *Characters of finite Coxeter groups and Iwahori-Hecke algebras.*
- J.-P. Serre, *Cohomologie des groupes discrets.*
- T. Terragni & T.S. Weigel, *The Coxeter complex and the Euler characteristic of a Hecke algebra*, ArXiv:math/1110.4981 .