Symplectic alternating algebras

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Symplectic alternating algebras

1. Powerful 2-Engel 3-groups.
2. Symplectic alternating algebras.
3. Some general structure theory.
4. Nilpotence, solvability and nil-conditions.

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Questions(Caranti).

(a) Does there exist a finite 2-Engel 3-group of class 3 where Aut G = Aut$_c G \cdot$ Inn G? (Yes. Abdollahi, Linton and O'Brien (2010))

(b) Let G be a group for which every element commutes with all its endomorpic images. Is G nilpotent of class at most 2?

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Symplectic vector spaces play a role in the classification. For one isolated example of rank 5 the associated symplectic vector space turns out to have a richer structure. This leads us to Symplectic Alternating Algebras.

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Definition. Let F be a field. A symplectic alternating algebra over F is a triple $(V, (\ ,\ ), \cdot)$ where V is a symplectic vector space over F with respect to a non-degenerate aternating form $(\ ,\ )$ and $\cdot$ is a bilinear and alternating binary operation on V such that

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Remark. The condition above is equivalent to $(u \cdot x, v) = (u, v \cdot x)$

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The associated symplectic alternating algebra $L(G)$. Let $L(G) = H/G^3$ where

$$[a, b]^3 = x^{9(\bar{a}, \bar{b})}$$

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$$\begin{aligned} u_i \cdot u_j &= \alpha_{ij}(1)u_1 + \cdots + \alpha_{ij}(2r)u_{2r} \\ (u_i, u_j) &= \beta_{ij} \\ &\Updownarrow \\ [h_i, h_j] &= h_1^{3\alpha_{ij}(1)} \cdots h_{2r}^{3\alpha_{ij}(2r)} x^{3\beta ij} \end{aligned}$$

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The map $L^3 \to F,\ (u, v, w) \mapsto (u \cdot v, w)$ is an alternating ternary form and that each alternating ternary form defines a unique symplectic alternating algebra. Classifying symplectic alternating algebras of dimension $2r$ over F is then equivalent to finding all the Sp(V) orbits of $\wedge^3 V$, under the natural action, where V is the symplectic vectorspace of dimension $2r$ with non-degenerate alternating form.

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Over the field $\mathbb{Z}_3$ there are 31 algebras of dimension 6 (T, 2008).

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Theorem 4. Either L contains an abelian ideal or L is semisimple. In the latter case the direct summands are uniquely determined as the minimal ideals of L

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Proposition 3.(Tota, Tortora, T) Let L be a symplectic alternating algebra that is abelian-by-(class c). We then have that L is nilpotent of class at most $2c + 1$.

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Proposition 4. (T^3) Suppose x is a left nil-element in L. Then we get a decompostion into a isotropic direct sum of standard x-invariant subspaces

$$L = W(a_1, b_1) \oplus_{\perp} \cdots \oplus_{\perp} W(a_n, b_n).$$

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Theorem 6.(T[3]) Let L be a symplectic alternating nil-2 algebra of dimension $2r$.

(a) If char $L \neq 2$ then L is nilpotent of class at most 3.

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The bounds in Theorem 6 are sharp.

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Classification.(T[3]) Symplectic alternating nil-algebras of dimension up to 8. (All nilpotent).