STEINBERG-LIKE REDUCIBLE CHARACTERS FOR CHEVALLEY GROUPS

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The Steinberg character plays outstanding role in the character theory of Chevalley groups. This character has many remarkable features. This inspires attempts to study, and possibly determine, the characters that have some, but not all, Steinberg character properties.

Let G be a simple Chevalley group in characteristic p, $|G|_p$ the p-part of the order of G. Then St_G, the Steinberg character of G,

(1) vanishes at all elements of order divisible by p and

(2) $St_G(1) = |G|_p$.

St_G is known to be a unique *irreducible* character with these properties. We drop the irreducibility condition to state the following

Problem. Determine reducible characters χ of G satisfying (1) and (2).

We call characters satisfying (1) and (2) above *Steinberg-like characters*.

The characters of projective modules satisfy (1). In 2008 Malle and Weigel determined Steinberg-like characters χ of projective modules assuming additionally that the trivial character 1_G is a constituent of χ. It turns out that the latter restriction can be dropped, and the following result can be viewed as completion of their project:

Theorem 1 [Z] Let χ be the character of a reducible projective module of degree $|G|_p$. Then 1_G is a constituent of χ, and therefore χ belongs to the list provided by Malle and Weigel.

The machinery of proving this result include the Brauer-Nesbitt correspondence between indecomposable projective modules and irreducible modules as well as Smith-Dipper results on Harish-Chandra restriction of irreducible modular representations of Chevalley groups.

The problem about Steinberg-like characters stated above looks much more ambitious than the analogues problem for projective modules solved in Theorem 1.

The experience obtained from this work suggests that the above problem could be solved by improving methods developed in [Z]. In this direction I work jointly with M Pellegrini.

Theorem 2 Groups of rank $r > 5$ have no Steinberg-like character.

More precisely:

Theorem 3 [Pelegrini-Z] Let G be a simple Chevalley group in characteristic p.

(1) Steinberg-like characters **do not exist** if G is one of the following groups:

$E_6(q), E_7(q), E_8(q), G_2(q)$;

$PSU_n(q), n > 3$, or $PSU_3(q)$ for $3 \not| q+1$;

$O^-_{2n}(q), n > 3$;

$PSL_n(q)$, $n > 5$, or $n = 4, 5$ and $3 \not| q+1$;

$PSp_{2n}(q), n > 2$, or $n = 2$ and $7 \not| q+1$,

$O^+_{2n}(q), n > 5$, or $O^+_{2n}(q)$, $n > 5$.

(2) Steinberg-like characters **do exist** if G is one of the groups:

$^2B_2(q)$, $^2G_2(q)$, $PSL_2(q)$, $PSL_3(q)$, $PSL_n(q)$ for $n = 4, 5$ and $3|q+1$, $PSU_3(q)$ for $3|q+1$, $PSp_4(q)$ for $7|q+1$.

The problem *remains open* for the groups

$O_{2n+1}(q)$ $n = 3, 4, 5$;

$O_{2n}^{+}(q)$ $n = 4, 5$;

$F_4(q), {}^2F_4(q), {}^3D_4(q)$,

but the work is in progress.

An essential feature of our results is that they are rather uniform with respect to q. In contrast, the characters of projective indecomposable modules depend on q. Even for the simplest case of $G = PSL_2(q)$ the characters of the projective indecomposable modules behave rather complex and irregular in terms of q.

One of the main tools in our analysis is the Harish-Chandra restriction/induction technique. Let L be a Levi subgroup of G. We keep the definition of a Steinberg-like character for L (one could keep this for arbitrary group too).

The initial step is to observe that if χ is a Steinberg-like character then so is the Harish-Chandra restriction $\overline{\chi}_L$ of χ to every Levi subgroup L of G. Let L' be the subgroup of L generated by the unipotent elements of L. Then restriction $\tilde{\chi}|_{L'}$ of $\overline{\chi}_L$ to L' is a Steinberg-like character of L'. Therefore, through induction assumption we get information on properties of $\tilde{\chi}|_{L'}$ for all Levi subgroups L of G.

We run induction on the BN-pair rank of G. So the groups of BN-pair rank 1 constitute the base of induction. Surprizingly, only the case with $G = SL_2(q)$ is straightforward. The other groups ${}^2B_2(q), {}^2G_2(q)$ and $SU_3(q)$ require considerable computational work. For projective module characters of degree $|G|_p$ this has been done manually by Malle and Weigel and by me in [Z]. Probably, the computations can be done manually in our cases too. But using the Chevie package of computer programs for Chevalley groups together with their character tables available for groups of small rank, we can significantly reduce manual computations.

The main problem is to perform the inductive step, and to convert information on $\tilde{\chi}|_{L'}$ to certain conclusion on χ. Our tools include general machinery of Harish-Chandra induction as exposed in the Curtis-Reiner book, in particular, the Benson-Curtis theorem, together with some basic theory of generalized Gelfand-Graev characters.

A generalized Gelfand-Graev character is defined to be ν^G, where ν is a linear character of a Sylow p-subgroup of G (where p is the defining characteristic of G). Then $(\chi, \nu^G) = 1$ for every Steinberg-like character χ. This allows us to take some control of the irreducible constituents of χ that are common with ν^G for some ν. Indeed, it is known that $\nu^G = \Gamma_L^G$, where L is a Levi subgroup of G and Γ_L is a usual Gelfand-Graev character of L.

Unfortunately, for groups other than $SL_n(q)$ there are irreducible characters that do not belong to any ν^G. We did not find any way to control such constituents of χ, and this is a source of certain difficulties. Strictly speaking, our non-computational results only concern with the "Gelfand-Graev part" χ_Γ of χ, where χ_Γ is the sum of all irreducible constituents of χ that occur in some ν^G. In many cases the equality $\chi = \chi_\Gamma$ holds, however, $\chi \neq \chi_\Gamma$ for $G = SU_3(q)$, ${}^2B_2(q)$, ${}^2G_2(q)$.

The formula $(\chi, \nu^G) = 1$ allows us to bound the multiplicities of the constituents common for χ and ν^G but does not help in identifying them. In particular, if G is of BN-pair rank 2 and χ is Steinberg-like character then the number of the constituents of χ in question is at most 4. However, even in these cases we cannot avoid computations with explicit character tables.

Results for groups of rank 2 allows to run induction by combining information obtained from complimentary (in a sense) Levi subgroups, however, only for groups $SU_n(q)$ and $D_n^-(q)$. For groups $G = SL_n(q)$ we are forced to perform computations with help of the Chevie package up to $n = 6$, to find out that $SL_6(q)$ has no *reducible* Steinberg-like character. Next by induction we arrive at a similar conclusion for $n > 6$. This result also yields this conclusion for groups that have Levi subgroups L with $L' \cong SL(6, q)$, in particular, E_6, E_7, E_8 and C_n, D_n, B_n with $n > 5$.

The induction step would be almost trivial if some Levi subgroup of G had no Steinberg-like character. In fact, this never happens, as all defect zero irreducible characters of L are Steinberg-like. However, in some cases of interest for every maximal Levi subgroup L every Steinberg-like character of degree $|G|_p$ is of defect 0. This leads to the natural question on whether it is true that a Steinberg-like character is irreducible if the Harish-Chandra restriction of it to every Levi subgroup is a sum of defect 0 irreducible characters. In the case of projective indecomposable modules the Brauer-Nesbitt correspondence reduces this to a similar question for irreducible modules, which are easy to handle. But in our case no analog of the Brauer-Nesbitt correspondence is available. This explain why one cannot mimic any machinery from the theory of projective modules to deal with Steinberg-like characters.

[1] G. Malle and Th. Weigel, *Finite groups with minimal* 1-*PIM*, Manuscripta Math. **126** (2008), 315 - 332.

[2] M. Pellegrini and A. Zalesski (in preparation)

[3] A. Zalesski, *Low dimensional projective indecomposable modules for Chevalley groups in defining characteristic* (submitted)