On words that are concise in residually finite groups

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dedicated to the memory of Brian Hartley

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Counterexample

$w(x, y) = [[x^{p^n}, y^{p^n}]^n, y^{p^n}]^n$ with n odd, $n \geq 10^{10}$ and p prime, $p > 5000$

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- (1966) Turner-Smith proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite.
- (1974) J. Wilson showed that every multilinear commutator word is concise (in the class of all groups).

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What about *powers* of multilinear commutator words?

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It remains unknown whether the word w^q is actually concise in the class of all groups.

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Note that all w-values in K have order dividing q.

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Since all w-values in our K have order dividing the prime-power q, we get that $w(K)$ is locally finite and so it is periodic.

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We deduce that $w(G)$ has finite exponent from the following key result.

Proposition

Let w be a multilinear commutator. There exist a (d, n, w)-bounded integer s and a (d, n)-bounded integer h with the following property:

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$$1 = T_1 \leq T_2 \leq \cdots \leq T_s = w(G)$$

such that every quotient T_i/T_{i-1} is an abelian subgroup generated by cyclic subgroups contained in $(G/T_{i-1})_w$, except possibly one quotient whose order is at most h.

Thank you!

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