The Burnside problem on periodic groups of odd exponents $n > 100$

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The Burnside problem on periodic groups (1902)

***Let** $a_1, a_2, \ldots, a_m$ **— be a finite sequence of independent elements generating a group** G, **in which for each element** x **the relation** $x^n = 1$ **holds, where n — is a given integer. Will the group defined this way be finite, and if yes, then what is its order**?*

Maximal periodic groups, presented by m generators and an identical relation $x^n = 1$,

$$\mathbf{B}(m, n) = \langle a_1, \ldots, a_m \mid x^n = 1 \rangle$$

are called **Free Burnside groups of period** n.

Finiteness of groups $B(m, n)$ was proved in the following cases:

- W. Burnside, 1902: $n \leq 3$,
- I. N. Sanov, 1940: $n = 4$,
- M. Hall, 1957: $n = 6$.

Finiteness was also proved for all finitely generated periodic *matrix* groups:

- W. Burnside in 1905 proved this for the case, when periods of all elements are uniformly bounded.
- I. Shur in 1911 proved the finiteness of all finitely generated periodic *matrix* groups.

W. Magnus on Burnside problem (W. Magnus, B. Chandler, The History of Combinatorial Group Theory, 1982, p. 47)

«**Turning now to papers in the theory of abstract groups which much later led to important developments, we must undoubtedly assign first place to a problem raised by Burnside in 1902.** Although there still remain many open questions, **at least the most difficult part of Burnside's problem has been solved: Adian and P. S. Novikov proved that** $B(m, n)$ **is infinite for all** $m \geq 2$ **and odd** $n \geq 665$ **(see Adian, 1975).**

This paper is possibly the most difficult paper to read that has ever been written on mathematics.»

W. Magnus on Burnside problem (W. Magnus, B. Chandler, The History of Combinatorial Group Theory, 1982, p. 47)

«**A comparison of the influence of Burnside's problem on combinatorial group theory with the influence of Fermat's last theorem on the development of algebraic number theory suggests itself very strongly.**
Both problems are of a rather special nature.

But in spite of their special nature, both problems have fascinated generations of mathematicians and have never been forgotten.»

W. Magnus on Burnside problem (W. Magnus, B. Chandler, The History of Combinatorial Group Theory, 1982, p. 154)

«Very much like "Fermat's last theorem" in number theory, Burnside's problem has acted as a catalyst for research in group theory.
The fascination exerted by a problem with an extremely simple formulation which then turns out to be extremely difficult has something irresistible about it to the mind of the mathematician.»

Novikov-Adian theorem

Negative solution of the Burnside Problem was first published by P.S. Novikov and S.I. Adian in joint papers: Novikov P. S., Adian S. I. "Infinite periodic groups I, II, III" . Izv. Akad. Nauk SSSR, Ser. matem., v. 32 , No. 1,2,3 (**1968**).

The main result of these papers is the following theorem.

Theorem 1. (**Novikov-Adian, 1968**). *Free periodic groups* $\boldsymbol{B}(m, n)$ *of odd periods* $\boldsymbol{n} \geq$***4381*** *are infinite.*

From this result obviously follows also infiniteness of free Burnside groups $\mathbf{B}(m, kn)$ for all exponents of the form kn with any even k.

"Restricted Burnside problem"(W. Magnus)

In 1950 from Burnside problem gemmated another problem, relating to finite periodic groups, that was formulated by W. Magnus in 1950. He called it «**Restricted Burnside problem**». Russian mathematician I. N. Sanov in his paper [1952] translated this term into Russian as «*Ослабленная проблема Бернсайда: "Weakened Burnside problem"* . This term can not be considered adequate, since it was based on Sanov's wrong conjecture that the full Burnside problem should have a positive solution. By this reason I prefer to call it in Russian as **Burnside-Magnus problem**, since it was formulated by W. Magnus and all obtained results so far are based on the approach outlined by W. Magnus and supported by many other authors.

"Restricted Burnside problem"(W. Magnus)

In the restricted Burnside problem, posed by W. Magnus, is required to check the existence of maximal finite group R$(m; n)$ for fixed period n and given number of generators m. If such a group exists, it is also required to find an estimation of the order of the group R$(m; n)$.

The two problems supplement each other. W. Magnus in his book calls them **Full Burnside problem and Restricted Burnside problem.**

Some historical facts happened at the Steklov Institute in 1959, that concerned to the Burnside Problem

1. A.I.Kostrikin in Spring of 1959 defended a doctoral thesis at the Steklov institute of mathematics.

2. A conversation between P.S.Novikov and V.M.Glushkov in this meeting.

3. A letter of prof. Kurosh to Novikov inviting him to give a plenary talk at the All-Union algebraic Conference at Moscow State university on the Burnside problem (in Summer of 1959).

4. A publication of Novikov's short paper in the journal "DAN USSR" (1959) with a declaration on his ideas related to a negative solution of the Burnside problem for exponents $n \geq 72$.

A continuation of the history in September, 1960

A temporary mini-seminar at the office of Prof. L.V.Keldysh, where Novikov gave some lectures on his plans to prove infiniteness of groups $B(m, n)$. Among participants were also students of P.Novikov – F.A.Kabakov and A.A.Fridman.

After a couple of lectures a conversations between Novikov and Adian on some technical questions related to a structure and a classification of periodic words moved to Novikov's appartment, that was near the Steklov institute, and continued in 1961. The other participants were out of this discussions at that time.

The International Mathematical Congress in Stockholm, 1962

An "additional"one-our lecture on the Burnside problem at a Restaurant after a joint dinner using a portable blackboard.

My dialog with Prof. S.Maclane by his initiative in the next day during an excursion on a ship.

Our decision to assume an additional restriction of oddness for exponents n in the end of 1962. This was after the Stockholm Congress.

Continuation of the history in 1963

In 1963 I have defended my doctoral thesis. Then P.Novikov suggested: we must declare officially that we work together in order to prove infiniteness of free Burnside groups for sufficiently large odd exponents before we finished the proof. His convincing arguments were the following: **If I spent all my time for that purpose we would have more chance to find a solution of the problem.**

One can recall that before our conversation P.Novikov accepted my proposal to restrict our plans by a case of odd exponents as some important assertions in our plans turned out to be not true for even exponents. We also agreed to consider a simultaneous classification of all reduced words in given alphabet relative to so called "*deepness of periods*" which we named **a rank**. Then I agreed to be considered as **a potential co-author of the future complete proof. Beginning Spring of 1963 we were meeting almost every day**.

Continuation of the history in 1964

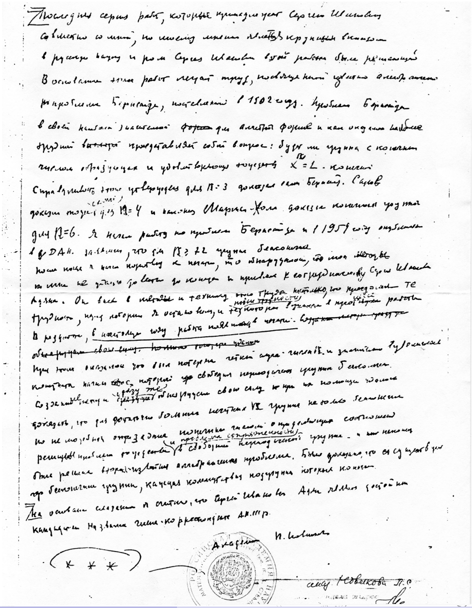
In 1964 P.S.Novikov officially announced our co-authorship and proposed to give a series of joint lectures at the Steklov institute in order to make clear the situation with our joint work. In Spring of 1964 we gave several lectures, where all necessary definitions and propositions with proofs were expounded in details for the first step of simultaneous induction (rank 1). **These lectures were given by myself in presence of P.S.Novikov.**

It was indicated in these lectures in 1964, that there are still some open questions, related to the inductive transition, but at that time we had no doubt about our real chances to overcome these difficulties in near future by increasing the value of odd exponent n.

The final of the complete proof, Summer of 1967

A complete proof of our final result was finished in Summer 1967. In September 1967 our joint papers were submitted for publication in "Izvestija" .

How P.S.Novikov describes this history.



Последняя серия работ, которые принадлежат Сергею Ивановичу
совместно со мной, по моему мнению является крупным вкладом
в русскую науку и роль Сергея Ивановича в этой работе была решающей.
В основании этих работ лежит труд, посвященный известной алгебраической
проблеме Бернсайда, поставленной в 1902 году. Проблема Бернсайда
в своей наиболее значительной [illegible] для алгебры форме и как оказалось наиболее
трудной [illegible] представляет собой вопрос: будет ли группа с конечным
числом образующих и удовлетворяющая тождеству $x^n = 1$ конечной.
Справедливость этого утверждения для $n = 3$ доказал сам Бернсайд. Санов
доказал то же (самое) для $n = 4$ и наконец Маршалл Холл доказал конечность группы
для $n = 6$. Я начал работу по проблеме Бернсайда и в 1959 году опубликовал
в ДАН заметку, что для $n \geq 72$ группы бесконечны.
Но когда я начал подготовку к печати, то обнаружилось, что мои методы
не [illegible] довести до конца и привлек к сотрудничеству Сергея Ивановича
Адяна. Он внес в методы и технику [illegible] настолько, что преодолел те
трудности, из-за которых я остановился, и те которые [illegible] в продолжении работы.
В результате, в настоящем году работа появилась в печати. [illegible]
[illegible]
При этом оказалось что была получена [illegible] числа n и значительно [illegible]
[illegible] что свободная периодическая группа бесконечна.
[illegible]
[illegible]
доказать, что для достаточно больших нечетных n группы не только бесконечны
[illegible]
[illegible] в свободной периодической группе [illegible]
[illegible] алгебраическая проблема. Было доказано, что существуют
[illegible] группы, каждая коммутативная подгруппа которых конечна.
На основании сказанного я считаю, что Сергей Иванович Адян вполне достоин
кандидатуры на звание члена-корреспондента АН СССР.

П. Новиков

Академик

***

подп. Новикова П. С.

P. S. Novikov's handwritten commentary to the history of our joint work

The last series of papers by Sergei Ivanovich and me is in my opinion a great contribution to Russian science, and the role of Sergei Ivanovich in this work was decisive. The cornerstone of the whole series is the work devoted to the famous problem posed by Burnside in 1902. The form of the Burnside problem that is most important in algebra and at the same time
has turned out to be the most difficult question: whether any group presented by a finite number of generators and satisfying an identity $x^n = 1$ ***is finite.***

The correctness of this assertion for n = 3 was proved by Burnside himself. Sanov proved the same for n = 4, and finally, Marshall Hall proved the finiteness of the groups for n = 6".

P. S. Novikov's handwritten commentary

I began working on the Burnside problem and in 1959 published in Doklady Akad. Nauk SSSR an announcement that the groups are infinite for $n \geq 72$. Later, when I began preparing for publication, it turned out that I could not drive my methods to the end, and I invited Sergei Ivanovich Adian to collaborate with me.

He brought so much in new methods and techniques for the work that he overcame those difficulties which had stopped me, along with the new difficulties which arose as we progressed. As a result the work was published this year.

P. S. Novikov's handwritten commentary

It turned out that we lost the case of even periods *n* and increased considerably the value of the constant beginning with which the free periodic group is infinite.
The methods created immediately demonstrated their power, and with their help we were able to prove that for sufficiently large odd *n* the group not only is infinite, but cannot be presented by finitely many defining relations. The word problem and the conjugacy problem for these free periodic groups were solved as well. Finally, another well-known algebraic problem was solved: we proved that there are infinite groups with only finite commutative subgroups.

Academician P. Novikov

On the proof

For the proof of this fundamental result the authors constructed a new theory of transformation for periodic words in groups $\mathbf{B}(m, n)$. A new theory was constructed from scratch, since no serious research on the same subject has been done before. It was known only the following more or less obvious lemma

Lemma 1. If two occurrences of periodic subwords A^r and B^t with minimal periods A and B in the given word W have common part of length $\geq$ of the length of AB, then A and B coincide, i.e. these subwords are parts of a longer subword with common period $A = B$.

On the proof

The solution of the Burnside problem in our joint paper was based on a classification of periodic words in a given alphabet, introduced by a simultaneous induction on a natural parameter α called *rank*. It was used to construct an appropriate system of defining relations for the group $\mathbf{B}(m, n)$.

In a new version of the proof we assume that an odd number $n \geq 101$ is fixed and consider reduced words in the alphabet of group $B(m, n)$.
Let

$$p \geq 6, \ q = 3p, \ n \geq 5q + 2p - 1.$$

$|W|$ is the length of a word W,

$W_1 = W_2$ means: W_1 and W_2 are two copies of the same word.

>From the presentation of the group $\mathbf{B(m, n)}$ immediately follows

Lemma 2. Any element x of group $\mathbf{B(m, n)}$ can be presented by some reduced word X which contains no periodic subword of the form $A^{\frac{n+1}{2}}$.

The following simple lemma is **a cornerstone** of our proof.

Lemma 3. If $A^t A_1 = B^r B_1$, where A_1 and B_1 are prefixes of A and B, and $|A^t A| \geq |A| + |B|$, then A и B both are powers of some word D. Hence if A и B are minimal periods, they coincide.

This lemma in origin has a short proof, but in our simultaneous induction we should generalize it for arbitrary rank α.

The monograph published in 1975

The first improved version of Novikov-Adian theory was presented in my monograph "**The Burnside problem and identities in groups** published in 1975, (Moscow, Nauka) (English transl., Springer-Verlag, 1979).
In this book Novikov-Adian theorem was proved for odd $n \geq 665$.

Theorem 2. (**Adian, 1975**). *Free periodic groups* $\boldsymbol{B}(m, n)$ *of odd periods* $\boldsymbol{n} \geq \boldsymbol{665}$ *are infinite.*

The monograph published in 1975

In the book published in 1975 the original Novikov-Adian theory has been generalized.
This generalization allowed to use the theory as a new powerful method to construct various groups with given properties.

In the generalised version groups also are obtained by adding new relations step by step and using a complicated simultaneous induction. But in the generalised version of the theory the relations may be not related to free Burnside groups $\mathbf{B}(\mathbf{m}, \mathbf{n})$. Moreover, these relations may be also not periodic.

John Britton's proof, 1973, 281 pages

In 1973 a British mathematician John Britton published a paper of 281 pages where he proved the following version of our theorem.

Theorem 3. (**J. Britton, 1973**). *There exists an odd number* N *such that the free periodic group* $\boldsymbol{B}(m, n)$ *is infinite for any odd period* $n \geq N$.

A.Yu.Olshanskii's papers, 1979, 1980

In his paper 1. "An infinite simple Noetherian group without torsion" (Izv. Akad. Nauk SSSR Ser. Mat., 43:6 (1979), 1328–1393) A. Yu. Ol'shanskii describes how he is using ideas from Novikov-Adian theory in the following words:

"Another feature of my paper is a proof of a large number of Lemmas by complicated simultaneous induction on the natural parameter i, which, following to the work [6], we call the rank. Furthermore, as in [6], the construction of the group is carried out by the addition of defining relations of new ranks, and periodic words play here an important role. For more complex notions it is used the terms which were introduced by S.I.Adian and P.S.Novikov [6]."

On Ol'shanskii's simplified proof of Novikov-Adian theorem

After another 3 years in his paper:
2. On the Novikov–Adyan theorem, Mat. Sb. (N.S.), 118(160):2(6) (1982), 203–235 Ol'shanskii published an "alternative proof" of the following version of Novikov–Adyan theorem: **Theorem 4.** (**Ol'shanskii, 1982**). *F*ree periodic

groups $\mathbf{B}(m, n)$ of odd periods $\mathbf{n} \geq 10^{10}$ are infinite.
One can notice that in both papers the Bibliography containes no reference to my monograph that was published in 1975 and was a result of the authors 7 year work on the futher improvement and generalization of Novikov-Adyan theory. One can notice that the bybliography in both papers contained no references to the new version of Novikov-Adyan theory described in the monograph.

Even exponents, S.Ivanov's proof, 1994

In 1992 S.Ivanov and I.Lysenok almost simultaneously announced proofs of the infiniteness of the free Burnside groups $\mathbf{B}(m, n)$ for all sufficiently large exponents n. After a few years they published their results. In principle, there was no need for these announcements without a proof, because P.Novikov has announced this result in 1959. In my survey publications on the Burnside problem I also mentioned that there was no doubt that our approach could be adapted for a solution of the Burnside problem for any sufficiently large exponent. But we did not have a formal proof of it. First Ivanov proved in 1994 that the groups $\mathbf{B}(m, n)$ are infinite for all periods of the form $n = 2^9 k \geq 2^{48}$.(See S. V. Ivanov, "The free Burnside groups of sufficiently large exponents", Internat. J. Algebra and Comput. 4:1-2 (1994), 1–308.)

Even exponents, I.Lysionok's proof, 1996

In 1996 I. Lysenok published a proof that $\mathbf{B}(m, n)$ is infinite for any $n = 2^4 k \geq 8000$, his proof contains 202 pages in the English translation, while Ivanov's paper with a proof for periods $n = 2^9 k \geq 2^{48}$ contains 307 pages. The point is that while both authors used the concise language of van Kampen diagrams, the schemes of induction they used are different. Ivanov used the so-called 'simplified' scheme introduced by Ol'shanskii and described in his book, and Lysenok's proof is based on the scheme described in my book published in 1975. In his paper S.Ivanov was proving a new theorem, so he could not write in a *sketchy manner* as it was done in Olshanskii's "short proofs" of Novikov-Adian theorem.

Joint paper of Delzant and Gromov, 2008

Theorem 5. (**Delzant-Gromov, 2008**). *For any hyperbolic group H there exists an odd number N such that the free periodic factor group $\boldsymbol{B}(H, n)$ of the group H is infinite for any odd period $n \geq N$.* (In "Journal of Topology" , V. 1, No. 4, 804-836).

This theorem was proved earlier by Ol'shanskii. (See A.Yu. Ol'shanskii, "Periodic factor groups of hyperbolic groups", Math. USSR-Sb. 72:2 (1992), 519–541).

In Ol'shanskii's proof we know the value $N = 10^{10}$ for a case if H is a free group F_m. But in Delzant-Gromov paper no information is given for the value of N, even for a case if $H = F_m$. It remainds Britton's proof with an unknoun value of the exponent N

Coulon's paper in 2012, his notion **"moves"** and his **"criterion"** .

A survey paper in "Uspekhi" , 2010, v.65, No.5

A detailed survey paper on investigations on the Burnside Problem and on the Restricted Burnside problem one can find in the following paper by author: "**The Burnside problem and related topics**" (Russian Math. Surveys 65:5 805–855). In this survey one can find a lot of interesting information from a history of investigations on the Full Burnside Problem" and the Restricted Burnside Problem" .

Some additional interesting details from the history of investigations on the Burnside Problem was given in **my recent talk on the same subject given in May 13 2013 at the Steklov Mathematical Instute**. It was an invited talk in the conference dedicated to 150 anniversary of V.A.Steklov. In this talk some interesting details from the history of our joint work were published for the first time.

Recent result for odd exponents $n \geq 101$, 2013

Theorem 6. (**Adian, 2013**). *Free periodic groups* $\boldsymbol{B}(m, n)$ *of odd periods* $\boldsymbol{n} \geq \boldsymbol{101}$ *are infinite.*
In this talk I shall introduce a new simplified modification of Novikov-Adian theory that allows to give a shorter proof and stronger results for add exponents. The basic important change in this modification is related to so called *active occurrences of periodic words of rank* α into a given reduced words.
Only for active occurrences of periodic words $A^t A_1$ into a reduced word $PA^t A_1 Q$ 0f rank $\alpha - 1$ we consider so called q-**reversal of rank** α that has a form

$$PA^t A_1 Q \rightarrow PA^{-(n-t-1)} A_2^{-1} Q,$$

where $A = A_1 A_2$ is an elementary period of rank α.

Some technical details

Definition: A subword E of the word A^n is called periodic word with the period A if $|E| \geq 2|A|$.
Cyclic periodic words $\tilde{A}^n$, it is written on a cycle. **Minimal periods**.
Definition: A subword E of the word A^n is t-**power of rank 1 with a given period** A if $|E| \geq t|A|$.

Elementary periods of rank 1

Elementary periods of rank 1 (depend on parameter $p = 6$)
Definition: A minimal period A is called **an elementary period of rank 1** if no p-power with a shorter period B occurs in A^n.

The first approximation:

$$\mathbf{B_1(m,n) \rightleftharpoons \langle a_1, a_2, \ldots, a_m;\ \{A^n = 1\} \mid A \in El_1 \rangle},$$

where $\mathbf{El}_1$ is the set of all elementary periods of rank 1.

q-reversals of rank 1

A word X is called **restricted in rank 1** if and only if X is reduced and no elementary $(n-q)$-power of rank 1 occurs in X. $\mathbf{R}_1(n-q)$ = set of all 1-restricted words of rank 1.

q-reversals of rank 1 look as follows:

$$X = PA^tA_1Q \;\rightarrow\; P(A^{-1})^{n-t-1}A_2^{-1}Q = Y.$$

Normal forms of reduced words in rank 1

Active occurrences of q-powers and p-powers.

A system of kernels of rank 1 for a given word $X \in \mathbf{R}_1$.

Normal form in rank 1 for 1-restricted words.

$$X = u_1 E_1 u_2 E_2 ... u_i E_i u_{i+1} ... u_k E_k u_{k+1}$$

Operation of coupling in rank 1 and the group $\Gamma_1(m, n)$

The relation $X \sim_1 Y$ and the binary operation of **Coupling in rank 1** $[X, Y]_1 = Z$ defined for $X, Y \in \mathbf{R}_1(n-q)$:
$[X, Y]_1 = Z$ **if and only if** $\exists X_1(X_1 \sim_1 X)$ and $\exists Y_1(Y_1 \sim_1 Y)$ such that $[X_1, Y_1]_0 = Z \in R_1(n-q)$.

Definition of the group $\Gamma_1(m, n)$.

Theorem. The groups $\mathbf{B_1(m, n)}$ and $\Gamma_1(m, n)$ are isomorphic.

Exponential growth

It was proved in above mentioned monograph published in 1975 the group $B(r, n)$ has an exponential growth. The growth function is very close to one for absolutely free group of the same number of generators $r > 1$.

Random walks and non-amenability

Later, in 1982, a non-amenability of the groups $B(r, n)$ for odd $n \geq 665$ and $r > 1$ has been proved as well (solution of von Neuman problem). It is the only known case of a non-amenable group that is satisfying a nontrivial identity.

It was also proved that the symmetric random walk over these groups is transient (Harry Kesten problem).

The finite basis problem in Group theory

There are various group theoretic problems that don't apply to the periodic groups, but they also were solved by the author using the created method. We list some of them.

The first and very simple examples of infinite irreducible systems of group identities in two variables(1969) (a solution of the finite basis problem for group varieties, H. Neuman):

$$\{(x^{pn}y^{pn}x^{-pn}y^{-pn})^n = 1\},$$

where $n = 1003$ and p runs all primes.

Construction of an interesting torsion-free group $\mathbf{A}(m, n)$

A construction of finitely generated group $\mathbf{A}(r, n)$ that is a **noncommutative analogue of the additive group of rational numbers**, i.e. the groups with an infinite intersection of any two nontrivial subgroups (1971). This problem was formulated by P.G. Kontorovich from Sverdlovsk in 1938. For the required group $\mathbf{A}(r, n)$ one can consider a central extension of the group $B(r, n)$ by an infinite cyclic center generated by an element c. Adding to $\mathbf{A}(r, n)$ one more relation $c^n = 1$ we receive an example of finitely generated countable infinite group which admits only the discrete topology (A.A. Markov's problem).

The operation of n-periodic product of groups

III. New commutative and associative operations of product of groups, satisfying the hereditary property for subgroups, 1976 (A.I. Maltsev's problem). The simplicity criterion for the periodic products of groups (proved in 1978) gave an opportunity to construct many new classes of finitely generated infinite periodic simple groups. Specter of periodic group

A.Yu.Olshanskii's papers since 1979, 1982

1. A. Yu. Ol'shanskii, An infinite simple Noetherian group without torsion, Izv. Akad. Nauk SSSR Ser. Mat., 43:6 (1979), 1328–1393.

"Another feature of my paper is a proof of a large number of Lemmas by complicated simultaneous induction on the natural parameter i, which, following to the work [6], we call the rank. Furthermore, as in [6], the construction of the group is carried out by the addition of defining relations of new ranks, and periodic words play here an important role. For more complex notions it is used the terms which were introduced by S.I.Adian and P.S.Novikov [6]."

2. On the Novikov–Adyan theorem, Mat. Sb. (N.S.), 118(160):2(6) (1982), 203–235

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A gap in the first Kostrikin's proof and his book of 1986/1990.

In 1959 A. I. Kostrikin published a paper named «**On the Burnside problem**», where he was proving the existence of maximal finite group $R(m, n)$ for any prime n. Such a title gave a false impression, that this is a solution of the Burnside problem posed in 1902. As it became known later there was a significant gap in the proof of this result of A. I.Kostrikin published in 1959.

The complete proof of this result was set out in his monograph «Around Burnside», published in 1986 (the Englishe translation apeared in 1990).

Adian-Razborov constructive proof of Kostrikin's theorem

I was a referee of this book of Kostrikin. This proof also was conducted from contrary and did not give any estimation for the orders of groups $R(m, n)$.

Moreover, in his monograph, Kostrikin repeatedly noted, that he does not see a perspective for appearance of a direct proof that would give estimations for orders of groups $R(m, n)$.

The first constructive proof of Kostrikin's result, with an indication of primitive recursive upper estimations for the orders of the groups $R(m, n)$ was published in joint paper of author and A. A. Razborov 1987.

To eliminate confusion in terminology for the problem posed by W. Magnus and called «Restricted Burnside problem», we suggested a more adequate term **Burnside-Magnus problem** instead of unsuccessful term "Weakened Burnside problem"used by Kostrikin.

The publication of the first effective proof of Kostrikin's result on the existence of minimal finite periodic groups $R(m; n)$ for prime n with indication of primitive recursive estimation of orders of these groups was a jolt to intensify research on the Restricted Burnside problem.

In particular, soon appeared other effective proofs of this result, and then, in two papers 1990 and 1991 E. I. Zelmanov spread Kostrikin's result on the case, when n is any power of a prime number.

The general scheme of main concepts of Novikov-Adyan theory to study the Burnside groups $\mathbf{B}(m, n)$ for odd periods n

Let fix an odd number $n > 100$ and natural parameters p and q, where $6 = p << q << n$. By simultaneous induction on the natural parameter $\alpha \geq 0$, called *rank*, we define the following sets of words in the alphabet $a_1, a_2, ...a_m; a_1^{-1}, a_2^{-1}, ...a_m^{-1}$

$$P_\alpha, \ \mathcal{E}_\alpha, \ R_{\alpha,t} \ (p \leq t \leq n - p)$$

and the groups

$$\mathbf{B}_\alpha(m, n) = \langle a_1, a_2, \ldots, a_m | \ \{A^n = 1\} \mid (A \in \bigcup_{\beta \leq \alpha} \mathcal{E}_\beta)\} \rangle. \quad (1)$$

Let $R_{0,t}$ for any t be the set of all reduced words in the in the group alphabet and $P_0 = E_0 = \varnothing$.

Suppose that sets $P_\beta, \ \mathcal{E}_\beta$ and $R_{\beta,t}$ have been defined for any rank $\beta \leq \alpha$.

We set by definition

$X \in P_{\alpha+1} \Leftrightarrow X^3 \in R\alpha, 2q$ & (X has infinite order in $\mathbf{B}_\alpha(m,n)$) & (X is not equal in $\mathbf{B}_\alpha(m,n)$ to any word Y^t $t > 1$);

$X \in \mathcal{E}_{\alpha+1} \Leftrightarrow X \in P_{\alpha+1}$ & $\exists S, T(SX^3T \in R_{\alpha,q}$ & *and*($SX^3T$is not equal in $\mathbf{B}_\alpha(m,n)$) to any word $UY^pV \in R_{\alpha,q}$ for $Y \in P_{\alpha+1}$));

$X \in R_{\alpha+1,t} \Leftrightarrow X \in R_{\alpha,t}$ & (X is not equal in $\mathbf{B}_\alpha(\mathbf{m},\mathbf{n})$to any word $\mathbf{UY^{n-t}V} \in \mathbf{R}_{\alpha,\mathbf{t}}$ for $\mathbf{Y} \in \mathcal{E}_{\alpha+\mathbf{1}}$).

We call elements of $R\alpha, t$ *t-reduced words of rank* α, elements of P_α — *minimal periods of rank* α, and elements of $\mathcal{E}_\alpha$ —*elementary periods of rank* α.

All other concepts of the Novikov-Adian theory can be introduced on the basis of these concepts according definition given in the book [Adian, 1975], see I.4.

In particular, the following concepts are introduced and used. *Conditionally-periodic (integral) words of rank* $\alpha + 1$, *generating occurrence of rank* $\alpha + 1$, *supporting kernels of rank* α, *normal occurrences of rank* α and binary relation of *mutual normalizibility of two given normal occurrences of rank* α, *related periods of rank* $\alpha + 1$, *elementary periods and elementary words of rank* $\alpha + 1$, *maximal normal continuation of a given normal occurrence of elementary p-power of rank* $\alpha + 1$, *q-reversals of rank* $\alpha + 1$, *active occurrences in the given reduced word of rank* α and *the normal form in rank* $\alpha + 1$ *of the given reduced word of rank* α, *kernels of rank* $\alpha + 1$ and so on (see [Adian, 1975], I.4).

A large number of properties of these concepts is proved by unique simultaneous induction o n rank α. Among them we should mention the following two most important statements of the theory as the infiniteness of the group $\mathbf{B}(m, n)$ for sufficiently large odd n follows just from these 2 sentences.

Statement 1 (see [Adian, 1975], VI.1.2). *Every reduced word X is conjugate in some group $\mathbf{B}_\alpha(m, n)$) to a power A^r of an elementary period A of some rank β*

Statement 2 (see [Adian, 1975], IV.2.16). *If reduced words X and Y do not contain nonempty subwords of the form Z^q, and they are equal in $\mathbf{B}(m, n)$, then they coincide .*

It follows immediately from the Statement 1 that the set of relation

$$\{A^n = 1\} \mid (A \in \bigcup_{\alpha > 0} \mathcal{E}_\alpha)$$

gives a full system of defining relations for the Burnside group $\mathbf{B}(m, n)$.
It is easy to check that this is an independent system of defining relation for the group $\mathbf{B}(m, n)$.

The infiniteness of the Burnside group $\mathbf{B}(m, n)$ follows immediately from the Statement 2 and the existence of an infinite sequence of different squarefree words. The simplest example of such sequence of words in 2-generated group can be constructed as follows.

$C_0 = a_1, \;\; C_1 = a_1 a_2 a_1^{-1}$ and $C_{i+1} = C_i a_2 C_i^{-1}$ for any $i > 0$.