Model Theoretic Advances for Groups with Bounded Chains of Centralizers

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Ischia Group Theory
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Bounded Centralizer Chains

A group has **bounded chains of centralizers**, denoted $\mathfrak{M}_C$, if every chain of centralizers

$$1 < C_G(A_1) < C_G(A_2) < \ldots < C_G(A_n)$$

is finite. If there is a uniform bound d on such chains, then the least such $d \geq 1$ is the **centralizer dimension** of G, denoted $\dim(G)$. In this case G has **finite centralizer dimension** (fcd).

• $\mathfrak{M}_C$: Every centralizer is a centralizer of a finite subset.
• Centralizer dimension d: Every centralizer is a centralizer of a subset of size $\leq d$.

Ascending vs. descending chains does not matter because $C_G(C_G(C_G(A))) = C_G(A)$.

Stable groups

Model theorists care about $\mathfrak{M}_C$ groups because they contain the class of stable groups. All stable groups have fcd, in fact.

algebraic geometry	**stability**
algebraic subgroups	definable subgroups
polynomial equations	group equations (with quantifiers)
algebraic dimension	Morley rank
algebraic independence	forking

Examples of stable and $\mathfrak{M}_C$ groups.

- finite groups (stable)
- abelian groups (stable)
- torsion-free hyperbolic groups (stable)
- free groups (stable)
- algebraic groups over any field (stable if over alg. closed field)
- finitely generated nilpotent groups
- finitely generated metabelian groups
- finitely generated abelian-by-nilpotent groups
- polycyclic groups

Example of an unstable/ non-$\mathfrak{M}_C$ group: $G = \mathrm{Sym}(\mathbb{N})$.

What results about finite groups, abelian groups, algebraic groups, etc. generalize to stable groups?

Many results from algebraic groups extend to stable groups:

Theorem

If G is an algebraic group over an algebraically closed field and $H \leq G$ is a nilpotent subgroup, then $H \subseteq N$, an algebraic nilpotent subgroup of the same nilpotence class.

Theorem (Poizat)

If G is a stable group and $H \leq G$ is a nilpotent subgroup, then $H \leq N$, a definable nilpotent subgroup of the same nilpotence class.

Many results from finite groups extend to stable groups:

Theorem (Baer Suzuki)

If G is a finite group and for all $a, b \in G$, $\langle a, b \rangle$ is nilpotent, then G is nilpotent.

Theorem (stable Baer Suzuki)

If G is a stable group and for all $a, b \in G$, $\langle a, b \rangle$ is nilpotent, then G is locally nilpotent.
If there exists an $n < \omega$ such that for all $a, b \in G$, $\langle a, b \rangle$ is nilpotent of class at most n, then G is nilpotent.

Stable groups are closed under definable subgroup and definable quotient.

$\mathfrak{M}_C$ groups are closed under subgroup, but not quotient: $G/Z(G)$ could fail to be $\mathfrak{M}_C$ (Bryant)

So no nice induction arguments:

- Cannot quotient by the center to reduce nilpotence class.
- Cannot quotient by normal subgroups in hope of reducing the length of a chain of centralizers.

Both these induction ideas are key to many proofs for stable groups.

$\mathfrak{M}_C$ groups are also not nice logically (not axiomatizable). So many model theoretic tools unavailable.

Theorem ()

If G is a stable group, then the Fitting subgroup $F(G)$ is nilpotent.

Theorem (Derakhshan Wagner 1997; Bludov 1998)

If G is an $\mathfrak{M}_C$ group, then the Fitting subgroup $F(G)$ is nilpotent.

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Theorem (Wagner 1999)

In an $\mathfrak{M}_C$ group, $F(G)$ equals the set of bounded left Engel elements of G.

$$F(G) = \{a \in G \mid \forall b \in G\, [[\ldots[[b, a], a], \ldots, a] = 1\}$$

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Corollary ($\mathfrak{M}_C$ Baer Suzuki)

If G is a $\mathfrak{M}_C$ group and for all $a, b \in G$, $\langle a, b\rangle$ is nilpotent, then G is locally nilpotent.
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Ingredients used:

- [Yen, 1979] A locally nilpotent $\mathfrak{M}_C$ group is solvable.

- For $\mathfrak{M}_C$ groups, we have equivalence between
  1. local nilpotence
  2. hypercentrality
  3. normalizer condition

(2) $\Rightarrow$ (3) due to Bryant (1979)
(3) $\Rightarrow$ (1) due to Derakhshan and Wagner (1997)
(1) $\Rightarrow$ (2) due to Bludov (1998)

Theorem (Wagner 1999)

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$$F(G) = \{a \in G \mid \forall b \in G\, [[\ldots[[b,a],a],\ldots,a] = 1\}$$

Corollary (Wagner 1999)

In an $\mathfrak{M}_C$ group, $F(G)$ is definable (with no parameters).

Model theoretic result on definability. Can we get more results on definability?

Recall:

Theorem (definable envelopes)

If G is a stable group and $H \leq G$ is a nilpotent subgroup, then $H \leq D$, a definable nilpotent subgroup of the same nilpotence class.

Theorem (Altınel, B. 2011). In an $\mathfrak{M}_C$ group G, if H is a nilpotent subgroup of class n, then $H \leq D$, a definable subgroup of G that is also nilpotent of class n. Furthermore D is normalized by all elements that normalize H.

If G is fcd of dimension d, then D is uniformly definable with $n \cdot d$ parameters from H.

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Proof:?

- No quotients
- No elementary extensions (model theory tool)
- No Engel conditions
- YES, lots of Three Subgroups Lemma to make the most of our commutator identities.

Sketch of Proof

Generalize: For $P \leq G$, the **nth iterated centralizer** $C_G^n(P)$ of P is:

$$C_G^0(P) = 1$$
$$C_G^n(P) = \{g \in \bigcap_{k<n} N_G(C_G^k(P)) \mid [g, P] \subseteq C_G^{n-1}(P)\}.$$

Fact: For all $n < \omega$, $P \cap C_G^n(P) = Z_n(P)$, so if P is nilpotent of class n, then $P \leq C_G^n(P)$.

Sketch of Proof

Lemma (Altınel, B. 2011). If G is an $\mathfrak{M}_C$ group and $H \leq G$ any subgroup, then the nth iterated centralizer $C_G^n(H)$ is definable with parameters from H.

For each n, there is a uniform definition for $C_G^n(H)$ across groups of dimension d involving dn parameters.

Sketch of Proof

Theorem (Altınel, B. 2011). In an $\mathfrak{M}_C$ group G, if H is a nilpotent subgroup of class n, then $H \leq D$, a definable subgroup of G that is also nilpotent of class n. Furthermore D is normalized by all elements that normalize H.

First attempt: Given $H \leq G$ of nilpotence class n, then $C_G^n(H) \geq H$ and $C_G^n(H)$ is definable.

Problem: $C_G^n(H)$ need not be nilpotent. Could try $Z_n(C_G^n(H))$, but do not necessarily have $Z_n(C_G^n(H)) \geq H$.

Why? Have $[[[C_G^n(H), H], H], \ldots, H] = 1$ but need $[[[H, C_G^n(H)], C_G^n(H)], \ldots, C_G^n(H)] = 1$.

Sketch of Proof

Observe: $H \leq C_G(C_G(H))$. Second attempt: rather than iterate *centralizers*, iterate *centralizers of centralizers*. (this construction works for any subgroup H)

$$E_0 = G \quad \geq \quad E_1 = C_G(C_G(H)) \quad \geq E_2 \geq E_3 \geq \ldots \geq H$$

Sketch of Proof

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Since each E_k contains H, we can compute $C^n(H)$ *inside* E_k.

$$E_0 := G$$

$$E_{k+1} := \{g \in E_k \,|\, [g, C_{E_k}^{k+1}(H)] \leq C_{E_k}^{k}(H)\}$$

Example:
$E_1 = \{g \in E_0 = G \,|\, [g, C_G^1(H)] \leq C_G^0(H) = 1\} = C_G(C_G(H))$.

Highly nontrivial to show that E_k is definable (definition a priori uses all of H, but can show we need only finitely many elements from H).

Control more and more iterated centralizers at each step: for all $j \leq k$, get

$$C^j_{E_{k-1}}(E_k) = C^j_{E_{k-1}}(H)$$

Inductive step to prove this uses:

Lemma (Altinel, Baginski, building on Bryant (1979))

G an $\mathfrak{M}_C$ group, $X \leq P$ subgroups. If

1. $C_G(X) = C_G(P)$;
2. $C^i_G(X) = C^i_G(P)$ *for all* $i \in \{0, \ldots, k-1\}$;
3. $[\gamma_k(P), C^k_G(X)] = 1$

Then $C^k_G(X) = C^k_G(P)$.

to conclude $C^k_{E_{k-1}}(H) = C^k_{E_{k-1}}(E_k)$.

These steps guarantee that for all $n \leq k$

$$C^n_{E_k}(H) = Z_n(E_k)$$

Now if H is nilpotent of class n, we have

$$H = Z_n(H) = H \cap C^n_{E_n}(H) = H \cap Z_n(E_n) \leq Z_n(E_n)$$

so $Z_n(E_n)$ was our definable nilpotent envelope of H.

$$
\begin{array}{llllllll}
E_0\colon & 1 & \leq C^1_{E_0}(H) & \leq C^2_{E_0}(H) & \leq C^3_{E_0}(H) & \leq C^4_{E_0}(H) & \leq \dots \\
\cup & \shortparallel & \quad\cup & & & & \\
E_1\colon & 1 & \leq C^1_{E_1}(H) & \leq C^2_{E_1}(H) & \leq C^3_{E_1}(H) & \leq C^4_{E_1}(H) & \leq \dots \\
\cup & \shortparallel & \quad\cup & & & & \\
E_2\colon & 1 & \leq C^1_{E_2}(H) & \leq C^2_{E_2}(H) & \leq C^3_{E_2}(H) & \leq C^4_{E_2}(H) & \leq \dots \\
\cup & \shortparallel & \quad\cup & & & & \\
E_3\colon & 1 & \leq C^1_{E_3}(H) & \leq C^2_{E_3}(H) & \leq C^3_{E_3}(H) & \leq C^4_{E_3}(H) & \leq \dots \\
\cup & \shortparallel & \quad\cup & & & & \\
E_4\colon & 1 & \leq C^1_{E_4}(H) & \leq C^2_{E_4}(H) & \leq C^3_{E_4}(H) & \leq C^4_{E_4}(H) & \leq \dots \\
\cup & \shortparallel & \quad\cup & & & & \\
E_5\colon & 1 & \leq C^1_{E_5}(H) & \leq C^2_{E_5}(H) & \leq C^3_{E_5}(H) & \leq C^4_{E_5}(H) & \leq \dots
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E_1\colon & 1 & \leq Z_1(E_1) & \leq C^2_{E_1}(H) & \leq C^3_{E_1}(H) & \leq C^4_{E_1}(H) & \leq \dots \\
\cup & \shortparallel & \shortparallel & \cup & & & \\
E_2\colon & 1 & \leq Z_1(E_2) & \leq Z_2(E_2) & \leq C^3_{E_2}(H) & \leq C^4_{E_2}(H) & \leq \dots \\
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Continuations?

Theorem (Poizat) If G is a stable group, and N is a solvable subgroup of G, then N is contained in a definable solvable subgroup H of G of the same derived length as N.

True for $\mathfrak{M}_C$?

Needs more group theory, not just model theory.

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A solvable version of the Baer-Suzuki theorem (due to Guest (and others?)) exists for finite groups. True for $\mathfrak{M}_C$? Not even clear if true for stable.