The Lie module for the symmetric group

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The word algebra

We use symbols $1, 2, \ldots, n$ and consider words $i_1 i_2 \cdots i_r$ where $i_1, \ldots, i_r \in \{1, \ldots, n\}$. These form a semigroup under concatenation:

$$(i_1 \cdots i_r)(j_1 \cdots j_s) = i_1 \cdots i_r j_1 \cdots j_s.$$

Let F be a field. For simplicity later we take F to be algebraically closed. Let A denote the set of all formal F-linear combinations of words. Thus A is an algebra with multiplication coming from concatenation.
For $u, v \in A$ write $[u, v] = uv - vu$ and use 'left-normed' notation. For example,

$$[1, 2, 3] = [[1, 2], 3] = [12 - 21, 3] = 123 - 213 - 312 + 321.$$

We have the identities

$$[u, v, w] + [v, w, u] + [w, u, v] = 0, \quad [u, u] = 0, \quad [u, v] = -[v, u].$$

The symmetric group

Let $\sigma \in S_n$ where $r\sigma = i_r$ for $r = 1, \dots, n$. We use 'image notation', writing σ as a word:

$$\sigma = (1\sigma)(2\sigma)\cdots(n\sigma) = i_1 i_2 \cdots i_n.$$

Thus S_n is identified with a set of words in A.
The identity element e of S_n becomes $12\cdots n$.
Write $\circ$ for composition, the operation of S_n, so that $r(\sigma \circ \tau) = (r\sigma)\tau$ for $r = 1, \dots, n$.
If $\sigma = i_1 i_2 \cdots i_n$ in image notation then we have $r(\sigma \circ \tau) = i_r\tau$ so we have $\sigma \circ \tau = (i_1\tau)\cdots(i_n\tau)$ in image notation.
The group algebra FS_n consists of all formal F-linear combinations of elements of S_n with multiplication $\circ$ in FS_n coming from $\circ$ in S_n. FS_n has dimension $n!$. We can identify FS_n with a subset of A (but not a subalgebra).
Recall that FS_n is a right FS_n-module (the regular module) under multiplication $\circ$.

The Lie module defined

If $i_1, i_2, \ldots, i_n$ are distinct elements of $\{1, 2, \ldots, n\}$ then $[i_1, i_2, \ldots, i_n] \in FS_n \subset A$. Let $\mathrm{Lie}(n)$ denote the F-span of all such elements $[i_1, i_2, \ldots, i_n]$. Thus $\mathrm{Lie}(n) \subset FS_n \subset A$. It is easy to check that

$$[i_1, i_2, \ldots, i_n] \circ \tau = [i_1\tau, i_2\tau, \ldots, i_n\tau].$$

Thus $\mathrm{Lie}(n)$ is a right ideal of FS_n. In other words it is a submodule of the regular FS_n-module.
$\mathrm{Lie}(n)$ is called the **Lie module** for S_n. This module arises in the study of free Lie algebras as well as in parts of algebraic combinatorics and algebraic topology.
From the point of view of a group-theorist it is a simply-defined module for an important group S_n, and S_n has a much-studied representation theory. It is natural to study $\mathrm{Lie}(n)$.

Properties of the Lie module

Recall that $\mathrm{Lie}(n)$ is spanned by all $[i_1, i_2, \ldots, i_n]$ with $i_1, \ldots, i_n$ distinct. These are not linearly independent. For example, for $n = 3$,

$$[3, 2, 1] = -[2, 1, 3] - [1, 3, 2] = [1, 2, 3] - [1, 3, 2].$$

We can express each $[i_1, i_2, \ldots, i_n]$ as a linear combination of elements of the form $[1, j_2, \ldots, j_n]$. It turns out that the elements $[1, j_1, \ldots, j_n]$ are linearly independent. Hence

$$\dim \mathrm{Lie}(n) = (n-1)!.$$

In fact if S_{n-1} denotes the stabiliser of 1 in S_n we get

$$\mathrm{Lie}(n){\downarrow_{S_{n-1}}} \cong FS_{n-1}.$$

The Dynkin–Specht–Wever element

Using backward cycles $(i, \ldots, 2, 1)$ in S_n define $\omega_n \in FS_n$ by

$$\omega_n = (e - (n, \ldots, 2, 1)) \circ \cdots \circ (e - (3, 2, 1)) \circ (e - (2, 1)).$$

It can be proved that $\omega_n \circ \omega_n = n\omega_n$, so, if n is not divisible by the characteristic of F, $(1/n)\omega_n$ is an idempotent of FS_n. ω_n is called the Dynkin–Specht–Wever element. It turns out that

$$\mathrm{Lie}(n) = \omega_n \circ FS_n,$$

giving an explicit description of $\mathrm{Lie}(n)$ as a right ideal of FS_n.

Characteristic 0

Suppose that char $F = 0$. Klyachko (1974) showed that

$$\text{Lie}(n) \cong U \uparrow_{\langle \kappa \rangle}^{S_n},$$

where κ is the n-cycle $(1, 2, \ldots, n)$ of S_n and U is a faithful 1-dimensional module for $\langle \kappa \rangle$. Hence there is a formula for the character of Lie(n).

Recall that the irreducible modules of S_n are modules D^λ indexed by the partitions λ of n.

Klyachko (1974) showed that almost every D^λ occurs as a summand of Lie(n).

Kraskiewicz and Weyman (1987) showed that the multiplicity of D^λ in a decomposition of Lie(n) is equal to the number of standard tableaux of shape λ and of 'major index' congruent to 1 modulo n.

Characteristic p preliminaries

From now on we take char $F = p \neq 0$. Recall that every FS_n-module V decomposes (essentially uniquely up to isomorphism) as a direct sum of indecomposable components. An indecomposable is projective if and only if it is a component of the regular module FS_n and V is projective if and only if all its components are projective.

In general we can write $V = V^{\text{proj}} \oplus V^{\text{pf}}$ where V^{proj} is projective and V^{pf} is 'projective-free', i.e. has no nonzero projective summand. V^{proj} and V^{pf} are uniquely determined up to isomorphism.

If $p \nmid n$ then $(1/n)\omega_n$ is an idempotent and Lie(n) is a summand of FS_n, so Lie(n) is projective. This is similar to the characteristic 0 case. The main case to consider is where $p \mid n$.

Every FS_n-module V decomposes in a natural way as a sum of 'block' components: $V = \bigoplus_{B \in \mathcal{B}} V_B$. The principal block is the block B_0 such that $W = W_{B_0}$ where W is the 1-dimensional trivial module.

Block components

From the above we have $\text{Lie}(n) = \bigoplus_{B \in \mathcal{B}} \text{Lie}(n)_B$.

Theorem (Erdmann & Tan, 2011). Let B be a non-principal block of FS_n. Then $\text{Lie}(n)_B$ is projective.

Thus, when B is non-principal, $\text{Lie}(n)_B$ can be written as a direct sum of projective indecomposables.

Theorem (RMB & Erdmann, 2012). A formula (in terms of Brauer characters) for the multiplicity of each projective indecomposable FS_n-module as a component of $\text{Lie}(n)_B$ where B is any non-principal block.

Non-projective components

There remains the problem of understanding $\mathrm{Lie}(n)_{B_0}$. The Brauer character of $\mathrm{Lie}(n)$ is known because the character is known in characteristic 0. By the previous theorem we know the Brauer character of each $\mathrm{Lie}(n)_B$ where B is non-principal. Hence we know the Brauer character of $\mathrm{Lie}(n)_{B_0}$. We have

$$\mathrm{Lie}(n)_{B_0} = \mathrm{Lie}(n)_{B_0}^{\mathrm{proj}} \oplus \mathrm{Lie}(n)_{B_0}^{\mathrm{pf}}.$$

If we can determine $\mathrm{Lie}(n)_{B_0}^{\mathrm{pf}}$ we can find its Brauer character and deduce the Brauer character of $\mathrm{Lie}(n)_{B_0}^{\mathrm{proj}}$. Then, since projective modules are determined by their Brauer characters, we can (in principle) find $\mathrm{Lie}(n)_{B_0}^{\mathrm{proj}}$. Thus everything would follow from knowledge of $\mathrm{Lie}(n)_{B_0}^{\mathrm{pf}}$, the 'projective-free' part of $\mathrm{Lie}(n)_{B_0}$.

Note that, since $\mathrm{Lie}(n)_B$ is projective when B is non-principal, we have $\mathrm{Lie}(n)_{B_0}^{\mathrm{pf}} = \mathrm{Lie}(n)^{\mathrm{pf}}$.

The non-projective part

I report on work with Susanne Danz, Karin Erdmann and Jürgen Müller.

Reduction Theorem. The FS_n-module Lie(n) is known if we know the FS_{p^m}-module Lie(p^m) for each p-power p^m dividing n (where p is the characteristic of F). This reduces many problems about Lie(n) to the case where n is a power of p.

Therefore we look at the FS_{p^m}-module $\mathrm{Lie}(p^m)^{\mathrm{pf}}$.

By work of Erdmann & Schocker (2006), $\mathrm{Lie}(p)^{\mathrm{pf}}$ is indecomposable in characteristic p for all p.
For $p = 2$, it is not difficult to show that $\mathrm{Lie}(4)^{\mathrm{pf}} = \mathrm{Lie}(4)$ and this is indecomposable. By computer calculations, $\mathrm{Lie}(8)^{\mathrm{pf}}$ is indecomposable (of dimension 816).
For $p = 3$, by computer calculations, $\mathrm{Lie}(9)^{\mathrm{pf}}$ is indecomposable (of dimension 1683).

Question. Is $\mathrm{Lie}(p^m)^{\mathrm{pf}}$ always indecomposable in characteristic p?

Vertices and sources

For the small values of p^m considered above, the indecomposable module $\mathrm{Lie}(p^m)^{\mathrm{pf}}$ has vertex E_{p^m} where E_{p^m} is a maximal elementary abelian subgroup of S_{p^m} (of order p^m and acting regularly on $\{1, \ldots, p^m\}$). Further, $\mathrm{Lie}(p^m)^{\mathrm{pf}}$ has a source which is an endo-permutation module and $\mathrm{Lie}(p^m)^{\mathrm{pf}}$ is an endo-p-permutation module, as defined by Urfer.

Question. Are these facts true in general?