Conjugacy class sizes of finite groups

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Basic question:

Definition

Let G be a finite group and let $\mathrm{cs}(G)$ denote the set of conjugacy class sizes of G. So $\mathrm{cs}(G) = \{|x^G| : x \in G\}$.

What does the arithmetical data $\mathrm{cs}(G)$ tell us about the algebraic structure of G?

Moreover, which sets of natural numbers can occur as a set $\mathrm{cs}(G)$ for some finite group G?

Clearly we don't know everything:

Examples

(i) $\mathrm{cs}(G \times A) = \mathrm{cs}(G)$ when A is abelian.

(ii) Nilpotency cannot be determined, there exists a non-nilpotent group H with $\mathrm{cs}(H) = \{1, 2, 4, 5, 10, 20\}$.

(iii) Solubility cannot be determined (Navarro 2014).

But we do get some information:

Examples

(i) If there exists $|x^G| = p^\alpha$ for some prime p then G is not simple (Burnside 1904).

(ii) If the multiset of class sizes is given then nilpotency can be recognised (Cossey, Hawkes & Mann 1992).

(iii) If there exist 3 mutually coprime class sizes then G is not simple (Tchounikhin 1930).

(iv) If there exist 2 mutually coprime class sizes then G is not simple (Arad & Fisman 1987 - uses CFSG).

One, visually appealing, way to view this problem is to consider the following graphs.

Definition

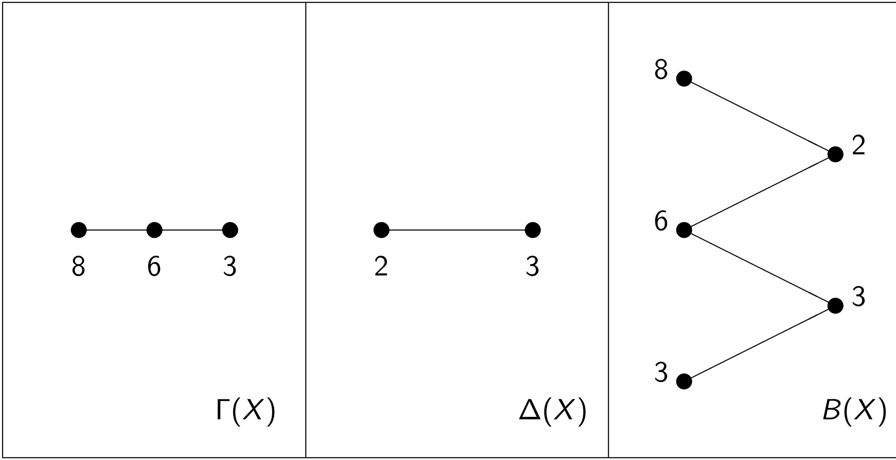
Let X be a set of positive integers.
(i) The *common divisor graph* of X has vertex set $X^* = X \setminus 1$ and an edge between $a, b \in X^*$ if a and b are not coprime. We denote the common divisor graph of X by $\Gamma(X)$.
(ii) The *prime vertex graph* has vertex set $\rho(X) = \bigcup_{x \in X} \pi(x)$ where $\pi(x)$ denotes the prime divisors of x. There is an edge between $p, q \in \rho(X)$ if pq divides x for some $x \in X$. The prime vertex graph is denoted by $\Delta(X)$.

The connection between these two graphs has been clarified by Iranmanesh & Praeger (2010), they defined the following bipartite graph $B(X)$.

Definition

The vertex set of $B(X)$ is given by the disjoint union of the vertex set of $\Gamma(X)$ and the vertex set of $\Delta(X)$, i.e. $X^* \cup \rho(X)$. There is an edge between $p \in \rho(X)$ and $x \in X^*$ if p divides x, i.e. if $p \in \pi(x)$.

Let $X = \{1, 3, 6, 8\}$.



Much is known when $X = \mathrm{cs}(G)$. The graphs have at most 2 connected components, and a connected component has diameter at most 3. The cases where there are 2 connected components or diameter 3 are known. Many authors have worked on these problems, Kazarin, Dolfi, Chillag, Herzog, Mann ...

Bubboloni, Dolfi, Iranmanesh and Praeger (2009) have investigated $B(\mathrm{cs}(G)) = B(G)$.

Taeri (2010) investigated when $B(G)$ is a cycle. He proved this only happens if $B(G)$ is a cycle of length 6 and then $G \cong SL_2(q) \times A$ where $q \in \{4, 8\}$ and A is abelian. He also proved that if $G/Z(G)$ is simple then $B(G)$ has no cycle of length 4 iff G as above.

Examples

For which groups does $B(G)$ have no cycles of length 4?

People have asked similar question using a different set of arithmetical data, namely the set of character degrees.
We denote by $\mathrm{cd}(G)$ the set $\{\chi(1) : \chi \in \mathrm{Irr}(G)\}$, where $\mathrm{Irr}(G)$ is the set of complex irreducible characters of G.
There are similarities:

Examples

(i) If G soluble then $\Delta(\mathrm{cd}(G))$ and $\Gamma(\mathrm{cd}(G))$ have at most 2 connected components, and the diameter is bounded by 3 (Manz, Willems & Wolf, 1985, 1987).

(ii) $\Delta(\mathrm{cd}(G))$ is a subgraph of $\Delta(\mathrm{cs}(G))$ (Casolo & Dolfi 2009).

and differences:

Examples

(i) Let p and q be distinct primes.
If $\mathrm{cs(G)} = \{1, p, q, pq\}$ then G is a direct product (Camina 1972). However, if $\mathrm{cd}(G) = \{1, p, q, pq\}$ then G is not necessarily a direct product (Lewis 1998).

(ii) If all conjugacy class sizes are square-fee then G is soluble (Chillag & Herzog 1990). But solubility does not follow from having square-free character degrees, eg. A_7.

Mark Lewis has considered the following condition.

Definition

A set of natural numbers X satisfies the *one prime hypothesis* if given $x, y \in X$ then either x and y are coprime or $\gcd(x, y)$ is a prime.

He proved for G soluble $|\mathrm{cd}(G)| \leq 9$ (2005) and with White (2007) for G insoluble $|\mathrm{cd}(G)| \leq 8$.

We extend the definition:

Definition

A set of natural numbers X satisfies the *one prime-power hypothesis* if given $x, y \in X$ then either x and y are coprime or $\gcd(x, y)$ is a prime power.

Then we have the following:

Definition

$B(G)$ has no cycles of length 4 iff $\mathrm{cs}(G)$ satisfies the one prime-power hypothesis.

• Note, the one prime power hypothesis is inherited by normal subgroups and quotients.

• If $C_G(x) < C_G(y)$ then $|y^G|$ is a prime power. Thus y lies in a soluble normal subgroup (Kazarin 1990). So, if G has no soluble normal subgroups and x and y are non-central with $C_G(x) \leq C_G(y)$, then $C_G(x) = C_G(y)$. Such a group is called an F-group, and they have been classified by Rebmann (1971).

Theorem

Let G satisfy the one prime power hypothesis. Let $S(G)$ be the maximal normal soluble subgroup of G. Then $G/S(G)$ is isomorphic to $SL_2(q)$ for $q \in \{4, 8\}$.

Theorem

Let G be a finite soluble group satisfying the one prime power hypothesis and $F(G)$ the Fitting subgroup of G. Then $G/F(G)$ is metabelian.

Suppose M/N is an abelian chief factor of G and let $\bar{G} = G/C_G(M/N)$. Let $1 \neq \bar{x}, \bar{y} \in \bar{G}$ with pre-images x and y. Let p divide M/N then p divides both $|x^G|$ and $|y^G|$. So $\gcd(|\bar{x}^{\bar{G}}|, |\bar{y}^{\bar{G}}|)$ is equal to 1 or p^a for some a. However if all elements of $\bar{G}$ have class sizes divisible by p then $O_p(\bar{G}) \neq 1$, a contradiction. Thus the conjugacy class sizes of $\bar{G}$ are either $\{1, m, n\}$ with m and n coprime or $\{1, m\}$. In both cases $\bar{G}$ is metabelian (Kazarin (1981), Itô (1953)). Finally, note that the intersection of the centralisers of the chief factors is precisely $F(G)$.