The group of inertial automorphisms of an abelian group

Ulderico Dardano (Napoli)
joint work with
Silvana Rinauro (Potenza)

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dedicated to Martin Newell

*on the occasion of his *-th birthday*

with many thanks to the organizers for given us the opportunity to give this talk in front of such an audience in this wonderful venue.

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B.H.Neumann's very celebrated Theorem (1955)

(FA) $\forall H \leq G \quad |H^G : H| < \infty \;\Leftrightarrow\; |G'| < \infty$ (finite-by-abelian).

J.T.Buckley, J.C.Lennox, B.H.Neumann, H.Smith, J.Wiegold, 1995

Let G be a group whose all periodic quotients are locally finite (this happens if G is hyper locally nilpotent-or-finite). Then

(CF) $\quad \forall H \leq G \quad |H/H_G| < \infty$ (core-finite, almost normal)

implies that G is abelian-by-finite, that is

(AF) $\quad \exists A \lhd G \quad : \quad A$ is abelian and G/A is finite.

Here H_G (resp. H^G) denote the largest (smallest) subgroup of G **invariant by G-conjugation** and contained in (containing) H.

How to put both pictures in the same framework?

FA is "stronger" than CF but does not imply CF.

Recall that subgroups H and K of a group G are **commensurable** iff their meet $H \cap K$ has finite index in both H and K i.e.

(C) $\quad |H : (H \cap K)| < \infty$ and $|K : (H \cap K)| < \infty$

This is an **equivalence** relation (introduced by Heineken-Specht, 1985).

Commensurability is a subgroup lattice congruence, provided $G = A$ is abelian.

A subgroup $H \leq G$ is said to be inert in G iff
H is commensurable to all its G-conjugates. that is

$$\forall g \in G, \forall H \leq G \quad |H : (H \cap H^g)| \cdot |H^g : (H \cap H^g)| < \infty$$

Both FA- and CF-groups G have all subgroups inert.

Notice that FA-groups have more: all subgroups are strongly inertial, i.e.:

$$\forall g \in G, \forall H \leq G, \quad |\langle H, H^g \rangle : (H \cap H^g)| < \infty$$

Recalls: TIN-groups

A group G is said totally inert (TIN) iff every subgroup is inert, that is $\forall g \in G, \forall H \leq G \quad |H : (H \cap H^g)| \cdot |H^g : (H \cap H^g)| < \infty$.
Note that all FC-groups are TIN (D.Robinson).

- D. Robinson (Ischia, 2006): **description of soluble TIN-groups** (under some finiteness conditions),

- V.V. Belayev, M. Kuzucuoğlu and E. Seckin (1999) +
M.R. Dixon, M.R. Evans, A. Tortora (2009):
there are no simple locally graded infinite TIN-groups,

Further, groups whose (finitely generated) subgroups are **"strongly inert"** have been recently investigated

- M.De Falco, F.de Giovanni, C. Musella, N. Trabelsi (2013)
Let G be any group.
If $\forall$ *finitely generated* $H \leq G \ \forall g \in G, \quad |\langle H, H^g \rangle : (H \cap H^g)| < \infty$,
then G is **(locally finite)-by-abelian**.

Say that an **Automorphism of an Abelian Additive group** A **is inertial** iff it maps each subgroups to a commensurable one.

$PAut(A)$ is the group of **power automorphisms**, that is the kernel of the setwise action of $Aut(A)$ on the lattice of subgroups of A.
To generalize this... Call **multiplication** of an abelian group A an automorphism acting by means of:
– p-adics on the primary p-components when A is periodic
– a rational number (on the whole of A) otherwise, where *multiplications by an non-integer rational are inertial iff the underlying group A has finite torsion free rank (**FTFR**)*.
$FAut(A)$ is the group of **finitary automorphisms** of A acting as the identity map on some finite index subgroup of A.
$IAut(A)$ is the kernel of the setwise action of $Aut(A)$ on the *quotient lattice* of classes of subgroups of A mod *commensurability*

$$PAut(A) \leq \textit{Above multiplications} \leq IAut(A) \geq FAut(A)$$

Recall that $\gamma \in Aut(A)$ is an almost-power automorphisms of an (abelian) group A if $\forall H \leq A \; |H : H_{\langle\gamma\rangle}| < \infty$.
Franciosi, de Giovanni, Newell (1995) showed that
- *almost-power automorphisms of an abelian group A form a group.*

Proposition, arXiv:1310.4625 If A has FTFR, then γ is inertial, provided $\forall H \leq A \;\; |H/(H \cap H^{\gamma})| < \infty$ (only).

Theorem, arXiv:1403.4193

If A is an *abelian* group then
$$IAut(A) = IAut_1(A) \times \{\pm 1\} \times F \qquad \text{where:}$$
- $IAut_1(A)$ is the group of inertial automorphisms acting trivially on $A/T(A)$
- F is free abelian with rank equal to the cardinality of the set of primes p s.t. A_p is bounded, A/A_p is p-divisible and either A has FTFR or A_p is finite.
- $IAut_1(A) \times \{\pm 1\}$ is the group of almost power automorphisms.

If A has not FTFR, we have $IAut_1(A) = FAut(A)$

the group $FAut(G)$ of finitary automorphism

Recall that $FAut(G)$ is the group of automorphisms γ of G acting as the identity map on some finite index subgroup of G i.e. $|G : C_G(\gamma)| < \infty$. Clearly $FAut(G) \leq IAut(G)$

THEOREMS

(Wehrfritz, 2002) if $G = A$ is abelian, $FAut(A)$ is locally finite.
(Belyaev-Shved, Ischia 2012) in the general case $FAut(G)$ is
- (locally finite)-by-abelian,
- locally (center-by-finite),
- abelian-by-(locally finite).

However $IAut(A)$ may contain non-periodic elements (e.g. p-adics).

PROBLEMs

Let $IAut(A)$ be the group of the inertial automorphisms of an abelian group A:
(LFA) is $IAut(A)$ (locally finite)-by-abelian ?
(ALF) is $IAut(A)$ abelian-by-(locally finite)?

Theorem B, [submitted] 2013, arXiv:1310.4625,

1) if $\Gamma = IAut(A)$ then $\Gamma' \leq FAut(A)$ is locally finite;
2) $IAut(A)$ is locally (center-by-finite).

Theorem, 2014, arXiv:1403.4193

There is a normal subgroup $\Gamma \leq IAut_1(A)$ s.t.:
i) $IAut(A)_1/\Gamma$ is locally finite;
ii) Γ acts by means of power automorphisms on its derived subgroup, which is a periodic abelian group.

Thus $IAut(A)$ is (metabelian and hypercyclic)-by-locally finite.

Finally: $IAut(\mathbb{Z}(p^\infty) \oplus \mathbb{Z})$ is NOT (locally nilpotent)-by-(locally finite), when $p \neq 2$.

If A/T is not finitely generated, it may happen that:
A has very few inertial automorphisms

$$IAut(\mathbb{Z}(p^\infty) \oplus \mathbb{Q}_p) = \{\pm 1\}$$

and also that $IAut(A)$ can be rather large

Theorem 2013 [submitted] & arXiv:1310.4625

The abelian group $IAut_1(A)/FAut(A)$ may contain an uncountable free subgroup.

Into details: there exists a countable abelian group A with FTFR such that there is $\Sigma \lhd IAut_1(A)$ with
- $\Sigma = St(A, T(A)) \simeq \prod_p \mathbb{Z}(p)$
-$\Sigma \cap FAut(A) = T(\Sigma) \simeq \bigoplus_p \mathbb{Z}(p)$,
where p ranges over the set of all primes.

Theorem, 2014, arXiv:1403.4193

Let A be a periodic abelian group, then

$$IAut(A) = PAut(A) \cdot (\Delta \cdot FAut(A))$$

where Δ is direct product of finite abelian groups.

Moreover, there is a set $\pi = \pi(A)$ of primes such that

$$\Delta \cdot FAut(A) = FAut(A_\pi) \times (\Sigma \rtimes \mathcal{I})$$

where $\Delta \leq \mathcal{I} \leq IAut(\Sigma)$ act faithfully by inertial automorphisms on the abelian π'-group Σ (which has bounded primary components).

Corollary: if A is periodic, then $IAut(A)$ is abelian-by-(locally finite).

Recall that if A has not FTFR, we have $IAut_1(A) = FAut(A)$

Proposition, 2014, arXiv:1403.4193

Let A be an abelian group with FTFR.
If either $A/T(A)$ finitely generated or $T(A)$ is bounded, then

$$IAut_1(A) = \Sigma \rtimes \Gamma_1$$

- $\Gamma_1 \simeq IAut(T)$ acts by means of inertial automorphisms on the periodic abelian group $\Sigma := St(A, T)$.
- $FAut(A) = \Sigma \rtimes \Phi_1$, where $FAut(T) \simeq \Phi_1 \leq \Gamma_1 \simeq IAut(T)$ acts faithfully by means of finitary automorphisms on Σ .

In particular, $IAut(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}) \simeq Hol(\mathbb{Z}_{p^\infty})$