The group of inertial automorphisms of an abelian group

Ulderico Dardano (Napoli)
joint work with
Silvana Rinauro (Potenza)
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dedicated to Martin Newell
on the occasion of his *-th birthday
with many thanks to the organizers for giving us
the opportunity to give this talk in front of such an audience
in this wonderful venue.

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B.H. Neumann’s very celebrated Theorem (1955)

(FA) \( \forall H \leq G \mid H^G : H \mid < \infty \Leftrightarrow \mid G' \mid < \infty \) (finite-by-abelian).


Let \( G \) be a group whose all periodic quotients are locally finite (this happens if \( G \) is hyper locally nilpotent-or-finite). Then

(CF) \( \forall H \leq G \mid H/H_G \mid < \infty \) (core-finite, almost normal)

implies that \( G \) is abelian-by-finite, that is

(AF) \( \exists A \triangleleft G : A \) is abelian and \( G/A \) is finite.

Here \( H_G \) (resp. \( H^G \)) denote the largest (smallest) subgroup of \( G \) invariant by \( G \)-conjugation and contained in (containing) \( H \).

How to put both pictures in the same framework?

FA is "stronger" than CF but does not imply CF.
Recall that subgroups \( H \) and \( K \) of a group \( G \) are **commensurable** iff their meet \( H \cap K \) has finite index in both \( H \) and \( K \) i.e.

\[
(C) \quad |H : (H \cap K)| < \infty \quad \text{and} \quad |K : (H \cap K)| < \infty
\]

This is an **equivalence** relation (introduced by Heineken-Specht, 1985). Commensurability is a subgroup lattice congruence, provided \( G = A \) is abelian.

A subgroup \( H \leq G \) is said to be **inert** in \( G \) iff \( H \) is commensurable to all its \( G \)-conjugates. that is

\[
\forall g \in G, \forall H \leq G \quad |H : (H \cap H^g)| \cdot |H^g : (H \cap H^g)| < \infty
\]

Both FA- and CF-groups \( G \) have all subgroups inert.

Notice that FA-groups have more: all subgroups are strongly inertial, i.e.: \( \forall g \in G, \forall H \leq G \), \( |\langle H, H^g \rangle : (H \cap H^g)| < \infty \)
A group $G$ is said **totally inert (TIN)** iff every subgroup is inert, that is $\forall g \in G, \forall H \leq G \ |H : (H \cap H^g)| \cdot |H^g : (H \cap H^g)| < \infty$.  

**Note that all FC-groups are TIN** (D. Robinson).

- D. Robinson (Ischia, 2006): **description of soluble TIN-groups** (under some finiteness conditions),
- V.V. Belayev, M. Kuzucuoğlu and E. Seckin (1999) + M.R. Dixon, M.R. Evans, A. Tortora (2009): there are no simple locally graded infinite TIN-groups,

Further, groups whose (finitely generated) subgroups are "**strongly inert"** have been recently investigated

- M.De Falco, F.de Giovanni, C. Musella, N. Trabelsi (2013)  
Let $G$ be any group. If $\forall$ finitely generated $H \leq G \ \forall g \in G$, $|\langle H, H^g \rangle : (H \cap H^g)| < \infty$, then $G$ is **(locally finite)-by-abelian**.
Say that an **Automorphism of an Abelian Additive group A** is **inertial** iff it maps each subgroups to a commensurable one.

\( \text{PAut}(A) \) is the group of **power automorphisms**, that is the kernel of the setwise action of \( \text{Aut}(A) \) on the lattice of subgroups of \( A \).

To generalize this... Call **multiplication** of an abelian group \( A \) an automorphism acting by means of:

- \( p \)-adics on the primary \( p \)-components when \( A \) is periodic
- a rational number (on the whole of \( A \)) otherwise, where **multiplications by an non-integer rational are inertial** iff the underlying group \( A \) has finite torsion free rank (**FTFR**).

\( \text{FAut}(A) \) is the group of **finitary automorphisms** of \( A \) acting as the identity map on some finite index subgroup of \( A \).

\( \text{IAut}(A) \) is the kernel of the setwise action of \( \text{Aut}(A) \) on the **quotient lattice** of classes of subgroups of \( A \) mod commensurability.

\[
\text{PAut}(A) \leq \text{Above multiplications} \leq \text{IAut}(A) \geq \text{FAut}(A)
\]
Recall that $\gamma \in Aut(A)$ is an almost-power automorphisms of an (abelian) group $A$ if $\forall H \leq A \ |H : H_{\langle \gamma \rangle}| < \infty$.
Franciosi, de Giovanni, Newell (1995) showed that
- *almost-power automorphisms of an abelian group $A$ form a group.*

**Proposition, arXiv:1310.4625** If $A$ has FTFR, then $\gamma$ is inertial, provided $\forall H \leq A \ |H/(H \cap H^\gamma)| < \infty$ (only).

**Theorem, arXiv:1403.4193**

If $A$ is an abelian group then

$$IAut(A) = IAut_1(A) \times \{\pm 1\} \times F$$

where:
- $IAut_1(A)$ is the group of inertial automorphisms acting trivially on $A/T(A)$
- $F$ is free abelian with rank equal to the cardinality of the set of primes $p$ s.t. $A_p$ is bounded, $A/A_p$ is $p$-divisible and either $A$ has FTFR or $A_p$ is finite.
- $IAut_1(A) \times \{\pm 1\}$ is the group of almost power automorphisms.

If $A$ has not FTFR, we have $IAut_1(A) = FAut(A)$
the group $F\text{Aut}(G)$ of finitary automorphism

Recall that $F\text{Aut}(G)$ is the group of automorphisms $\gamma$ of $G$ acting as the identity map on some finite index subgroup of $G$ i.e. $|G : C_G(\gamma)| < \infty$. Clearly $F\text{Aut}(G) \leq I\text{Aut}(G)$

**THEOREMS**

(Wehrfritz, 2002) if $G = A$ is abelian, $F\text{Aut}(A)$ is locally finite.

(Belyaev-Shved, Ischia 2012) in the general case $F\text{Aut}(G)$ is
- (locally finite)-by-abelian,
- locally (center-by-finite),
- abelian-by-(locally finite).

However $I\text{Aut}(A)$ may contain non-periodic elements (e.g. $p$-adics).

**PROBLEMS**

Let $I\text{Aut}(A)$ be the group of the inertial automorphisms of an abelian group $A$:

(LFA) is $I\text{Aut}(A)$ (locally finite)-by-abelian?

(ALF) is $I\text{Aut}(A)$ abelian-by-(locally finite)?

1) if $\Gamma = IAut(A)$ then $\Gamma' \leq FAut(A)$ is locally finite;
2) $IAut(A)$ is locally (center-by-finite).


There is a normal subgroup $\Gamma \leq IAut_1(A)$ s.t.: 

i) $IAut(A)_1/\Gamma$ is locally finite;

ii) $\Gamma$ acts by means of power automorphisms on its derived subgroup, which is a periodic abelian group.

Thus $IAut(A)$ is (metabelian and hypercyclic)-by-locally finite.

Finally: $IAut(\mathbb{Z}(p^\infty) \oplus \mathbb{Z})$ is NOT (locally nilpotent)-by-(locally finite), when $p \neq 2$. 

U. Dardano - S. Rinauro
The group of inertial automorphisms of an abelian group
The structure of $IAut(A)$ when $A$ is periodic


Let $A$ be a periodic abelian group, then

$$IAut(A) = PAut(A) \cdot (\Delta \cdot FAut(A))$$

where $\Delta$ is direct product of finite abelian groups.

Moreover, there is a set $\pi = \pi(A)$ of primes such that

$$\Delta \cdot FAut(A) = FAut(A_\pi) \times (\Sigma \wr \mathcal{I})$$

where $\Delta \leq \mathcal{I} \leq IAut(\Sigma)$ act faithfully by inertial automorphisms on the abelian $\pi'$-group $\Sigma$ (which has bounded primary components).

**Corollary:** if $A$ is periodic, then $IAut(A)$ is abelian-by-(locally finite).
Recall that if $A$ has not FTFR, we have $IAut_1(A) = FAut(A)$


Let $A$ be an abelian group with FTFR. If either $A/T(A)$ finitely generated or $T(A)$ is bounded, then

$$IAut_1(A) = \Sigma \rtimes \Gamma_1$$

- $\Gamma_1 \cong IAut(T)$ acts by means of inertial automorphisms on the periodic abelian group $\Sigma := St(A, T)$.
- $FAut(A) = \Sigma \rtimes \Phi_1$, where $FAut(T) \cong \Phi_1 \leq \Gamma_1 \cong IAut(T)$ acts faithfully by means of finitary automorphisms on $\Sigma$.

In particular, $IAut(\mathbb{Z}_{p}\infty \oplus \mathbb{Z}) \cong Hol(\mathbb{Z}_{p}\infty)$