The Theorems of Schur, Baer and Hall

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Thanks to Leonid Kurdachenko for his notes on this topic

Preliminaries

In 1904 I. Schur obtained the following result:

Theorem

Let G be a group and let $C \leq Z(G)$, the centre of G.Suppose that G/C is finite. Then G', the derived subgroup of G, is finite.

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A class $\mathfrak{X}$ of groups is called a Schur Class if

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Which classes of groups are Schur classes?

Examples

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These are examples of what we might even term "Universal Schur Classes"-the Schur type theorem holds for all groups.

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This result builds on earlier work of A. Lubotzky, A. Mann (finite case), S. Franciosi, F. de Giovanni, L. Kurdachenko (soluble case).

Example

- (A. Olshanskii) There is a group G such that $G = G'$; $Z(G)$ is free abelian of countable rank, and $G/Z(G)$ is an infinite p-group whose proper subgroups have order the prime p. ie. $G/Z(G)$ is a Tarski monster.

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- (S. I. Adian, 1971) There is a torsion-free group G such that $G/Z(G)$ is an infinite non-locally finite p-group of finite exponent.
- Thus the class of periodic groups is not a Schur class.

More generalizations

Let p be a prime. The group G has finite section p-rank r if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section of G precisely of order p^r.

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- The class of locally finite groups whose Sylow p-subgroups are Chernikov is a Schur class. (etc...)

Other directions

Let

$$1 = Z_0(G) \leq Z_1(G) = Z(G) \leq Z_2(G) \leq \cdots \leq Z_\alpha(G) \leq \dots$$

be the upper central series of G.
Let

$$G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \cdots \geq \gamma_\alpha(G) \dots$$

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- *(P. Hall, 1956) If $\gamma_{k+1}(G)$ is finite then $G/Z_{2k}(G)$ is finite*

Generalizations

- (M. De Falco, F. de Giovanni, C. Musella, Ya. P. Sysak, 2011) Let G be a group and let Z_α be the upper hypercentre of G. If G/Z_α is finite then G is finite-by-hypercentral.

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- (L. Kurdachenko, J. Otal, I. Ya. Subbotin, 2013) Assuming $|G/Z_\alpha| \leq t$ then there exists L such that $|L| \leq t^d$ where $d = \frac{1}{2}(\log_p t + 1)$, where p is the least prime divisor of t and G/L is hypercentral.

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- (L. Kurdachenko, J. Otal, 2013) Let G be a group and let Z_α be the upper hypercentre of G. If G/Z_α is a Chernikov group then G is Chernikov-by-hypercentral.

Other problems:Hegarty's Theorem

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$C_G(A) = \{g \in G | \alpha(g) = g$ for all $\alpha \in A\}$,
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Theorem

(Hegarty, 1994) If $G/C_G(\mathrm{Aut}\, G)$ is finite then $[G, \mathrm{Aut}\, G]$ is finite. In this case Aut G is also finite.

Note that $C_G(A)$ need not be normal in G. However this is the case if Inn $G \leq A$.

Results

Theorem

(MD, L. Kurdachenko, A. Pypka, 2014)
Let G be a group and let $\mathrm{Inn}\,G \leq A \leq \mathrm{Aut}\,G$. Suppose that $|A/\mathrm{Inn}\,G| \leq k$ and $|G/C_G(A)| \leq t$. Then $|[G, A]| \leq kt^d$, where $d = \frac{1}{2}(\log_p t + 1)$, and p is the least prime dividing t.

Results

Sketch proof Note that $C_G(A) \leq Z(G)$ so $|G/Z(G)| \leq t$. By Schur, Wiegold theorems $|G'| \leq t^m$ where $m = \frac{1}{2}(\log_p t - 1)$, with p least prime dividing t.

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$$[G, A]G'/G'| = |\sum_{1 \leq j \leq k} [G_{ab}, \bar{\alpha}_j]| \leq kt.$$

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Then $|[G, A]| \leq tk \cdot t^m$.

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$$[G, A]G'/G'| = |\sum_{1 \leq j \leq k} [G_{ab}, \bar{\alpha}_j]| \leq kt.$$

Then $|[G, A]| \leq tk \cdot t^m$. Is it possible to omit the hypothesis $A/\mathrm{Inn}\, G$ finite at the outset?

The Upper A-central series

Let $Z_1(G,A) = C_G(A)$ (normal, A-invariant). The Upper A-central series is

$$1 = Z_0(G,A) \leq Z_1(G,A) \leq Z_2(G,A) \leq \cdots \leq Z_\alpha(G,A) \leq \ldots,$$

where $Z_{\nu+1}(G,A)/Z_\nu(G,A) = Z_1(G/Z_\nu(G,A), A/C_A(Z_\nu(G,A))$. The usual condition holds for limit ordinals.The last term of this series, is the upper A-hypercentre.

The Lower A-central series

The lower A-central series of G is the descending, normal A-invariant series

$$G = \gamma_1(G, A) \geq \gamma_2(G, A) \geq \cdots \geq \gamma_\nu(G, A) \geq \gamma_{\nu+1}(G, A) \geq \ldots$$

where $\gamma_2(G, A) = [G, A]$ and $\gamma_{\nu+1}(G, A) = [\gamma_\nu(G, A), A]$. Limit ordinals treated as usual. The last term $\gamma_\delta(G, A) = \gamma_\infty(G, A)$ is the lower A-hypocentre of G.

Extension of Baer's Theorem

Theorem

(MD, L. Kurdachenko, A. Pypka, 2014)
Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $|A : Inn(G)| = k$ is finite. Let $Z_\alpha(G, A) = Z$ be the upper A-hypercentre of G. Suppose that $\alpha = m$ is finite and that G/Z is finite of order t. Then

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(i) $|\gamma_{m+1}(G, A)| \leq \beta(k, m, t)$, *for some function* β;

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(i) $|\gamma_{m+1}(G, A)| \leq \beta(k, m, t)$, *for some function* β;
(ii) $|\gamma_\infty(G, A)| \leq \beta_1(k, t)$, *for some function* β_1.

Extension of Hall's Theorem

Theorem

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Let G be a group and let A be a subgroup of $Aut(G)$ such that $Inn(G) \leq A$ and $|A : Inn(G)| = k$ is finite. If $\gamma_{m+1}(G, A)$ is finite of order t for some positive integer m, then $G/\zeta_{2m}(G, A)$ is finite of order at most $\eta(k, m, t)$, for some function η.

Linear Version

Let G be a group, F a field , A a right FG–module. The FG-centre of A is

$$Z_{FG}(A) = \{a \in A \mid a(g-1) = 0, g \in G\} = C_A(G).$$

Let ωFG be the augmentation ideal of the group ring FG, the two-sided ideal generated by the elements $g-1$ and let $A(\omega FG)$ be the derived submodule of A.

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G has finite secton 0-rank r if every torsion-free abelian section has rank at most r and there is such a section of rank r. Write $r_0(G) = r$.

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Theorem

(MD, L. Kurdachenko, J. Otal, 2013) Let $G \leq GL(F, A)$. Suppose that codim$_F\, Z_{FG}(A) \leq c$. If p is 0 or a prime and if $r_p(G) = r < \infty$ then dim$_F A(\omega FG) \leq \kappa(c, r)$, for some function κ, where p is the characteristic of F.

Example

Let A have countably infinite dimension over F and let $\{a_n \mid n \geq 1\}$ be a basis of A. Define $g_k \in GL(F, A)$ by

$$a_n g_k = \begin{cases} a_1 + a_k, & \text{if } n = 1; \\ a_n, & \text{if } n > 1. \end{cases}$$

Write $G = \langle g_k | k \in \mathbb{N} \rangle = \mathrm{Dr}_{k \geq 1} \langle g_k \rangle$.
char $F = 0$ implies G is free abelian.
char $F = p$ implies G is elementary abelian.
Then $\mathrm{codim}_F Z_{FG}(A) = 1$ but $A(\omega FG)$ is also the subspace generated by $\{a_n \mid n > 1\}$, so that $A(\omega FG)$ is infinite dimensional.

Upper and Lower *FG*-central series

Let $Z^0_{FG}(A) = 0, Z^1_{FG}(A) = Z_{FG}(A)$ and for all ordinals α set $Z^{\alpha+1}_{FG}(A)/Z^{\alpha}_{FG}(A) = Z_{FG}(A/Z^{\alpha}_{RG}(A))$, usual convention for limit ordinals. Obtain

$$0 = Z^0_{FG}(A) \leq Z^1_{FG}(A) \leq Z^2_{FG}(A) \leq \cdots \leq Z^{\alpha}_{FG}(A) \leq \cdots \leq Z^{\gamma}_{FG}(A)$$

The last term $Z^{\gamma}_{RG}(A)$ of this series is called the upper *FG*-hypercentre of A. .
Let $A = \gamma^1_{FG}(A)$ and $\gamma^2_{FG}(A) = A(\omega FG)$. Let $\gamma^{\alpha+1}_{FG}(A) = \gamma^{\alpha}_{FG}(A)(\omega FG)$ for all ordinals α, usual convention for limit ordinals. Obtain

$$A = \gamma^1_{FG}(A) \geq \gamma^2_{FG}(A) \geq \cdots \geq \gamma^{\alpha}_{FG}(A) \leq \gamma^{\alpha+1}_{FG}(A) \geq \cdots$$

Baer, Hall Theorems

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