The Theorems of Schur, Baer and Hall

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Thanks to Leonid Kurdachenko for his notes on this topic
In 1904 I. Schur obtained the following result:

**Theorem**

Let $G$ be a group and let $C \leq Z(G)$, the centre of $G$. Suppose that $G/C$ is finite. Then $G'$, the derived subgroup of $G$, is finite.
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Which classes of groups are Schur classes?
Examples

- If $G/Z(G)$ is a finite $\pi$-group, for some set of primes $\pi$, then $G'$ is a finite $\pi$-group.
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- (J. Wiegold, 1965) If $|G/Z(G)| \leq t$ then $|G'| \leq t^m$ where $m = \frac{1}{2} (\log_p t - 1)$ and $p$ is the least prime dividing $t$. 
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- (Ya D. Polovitzkii, 1964) If $G/Z(G)$ is Chernikov then $G'$ is Chernikov.
If \( G/Z(G) \) is a finite \( \pi \)-group, for some set of primes \( \pi \), then \( G' \) is a finite \( \pi \)-group.

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These are examples of what we might even term “Universal Schur Classes”-the Schur type theorem holds for all groups.
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This result builds on earlier work of A. Lubotzky, A. Mann (finite case), S. Franciosi, F. de Giovanni, L. Kurdachenko (soluble case).
(A. Olshanskii) There is a group $G$ such that $G = G'$; $Z(G)$ is free abelian of countable rank, and $G/Z(G)$ is an infinite $p$-group whose proper subgroups have order the prime $p$. i.e. $G/Z(G)$ is a Tarski monster.
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- Hence the class of groups with min (or max) is not a Schur class. And the class of groups of finite rank is not a Schur class.

- **(S. I. Adian, 1971)** There is a torsion-free group $G$ such that $G/Z(G)$ is an infinite non-locally finite $p$-group of finite exponent.
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(S. I. Adian, 1971) There is a torsion-free group $G$ such that $G/Z(G)$ is an infinite non-locally finite $p$-group of finite exponent.

Thus the class of periodic groups is not a Schur class.
Let $p$ be a prime. The group $G$ has finite section $p$-rank $r$ if every elementary abelian $p$-section of $G$ is finite of order at most $p^r$ and there is an elementary abelian $p$-section of $G$ precisely of order $p^r$. 
More generalizations

Let \( p \) be a prime. The group \( G \) has finite section \( p \)-rank \( r \) if every elementary abelian \( p \)-section of \( G \) is finite of order at most \( p^r \) and there is an elementary abelian \( p \)-section of \( G \) precisely of order \( p^r \).

(A. Ballester-Bolinches, S. Camp-Mora, L. Kurdachenko, J, Otal, 2013) Let \( G \) be locally generalized radical and suppose that \( G/Z(G) \) has section \( p \)-rank at most \( s \), for the prime \( p \). Then \( G' \) has section \( p \)-rank at most \( f(s) \).
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The class of locally finite groups whose Sylow $p$-subgroups are Chernikov is a Schur class. (etc...)
Let

\[ 1 = Z_0(G) \leq Z_1(G) = Z(G) \leq Z_2(G) \leq \cdots \leq Z_\alpha(G) \leq \cdots \]

be the upper central series of \( G \).

Let

\[ G = \gamma_1(G) \geq \gamma_2(G) = G' \geq \cdots \geq \gamma_\alpha(G) \cdots \]

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**Theorem**

(R. Baer, 1952) If \( G/Z_k(G) \) is finite, for some natural number \( k \), then \( \gamma_{k+1}(G) \) is finite.
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**Theorem**

- **(R. Baer, 1952)** If $G/Z_k(G)$ is finite, for some natural number $k$, then $\gamma_{k+1}(G)$ is finite.
- **(P. Hall, 1956)** If $\gamma_{k+1}(G)$ is finite then $G/Z_{2k}(G)$ is finite.
(M. De Falco, F. de Giovanni, C. Musella, Ya. P. Sysak, 2011) Let $G$ be a group and let $Z_\alpha$ be the upper hypercentre of $G$. If $G/Z_\alpha$ is finite then $G$ is finite-by-hypercentral.
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(L. Kurdachenko, J. Otal, I. Ya. Subbotin, 2013) Assuming $|G/Z_\alpha| \leq t$ then there exists $L$ such that $|L| \leq t^d$ where $d = \frac{1}{2}(\log_p t + 1)$, where $p$ is the least prime divisor of $t$ and $G/L$ is hypercentral.
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(L. Kurdachenko, J. Otal, 2013) Let $G$ be a group and let $Z_\alpha$ be the upper hypercentre of $G$. If $G/Z_\alpha$ is a Chernikov group then $G$ is Chernikov-by-hypercentral.
Let Aut\(G\) denote the automorphism group of \(G\), \(A \leq \text{Aut}\, G\).

**Theorem (Hegarty, 1994)** If \(G/C_G(\text{Aut}\, G)\) is finite then \(\left[G, \text{Aut}\, G\right]\) is finite. In this case \(\text{Aut}\, G\) is also finite.

Note that \(C_G(\text{Aut}\, G)\) need not be normal in \(G\). However this is the case if \(\text{Inn}\, G \leq A\).
Other problems: Hegarty’s Theorem

Let $\text{Aut } G$ denote the automorphism group of $G$, $A \leq \text{Aut } G$

$C_G(A) = \{ g \in G | \alpha(g) = g \text{ for all } \alpha \in A \},$

$[G, A] = \langle g^{-1} \alpha(g) | g \in G, \alpha \in A \rangle.$
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In this case $\text{Aut } G$ is also finite.

Note that $C_G(A)$ need not be normal in $G$. However this is the case if $\text{Inn } G \leq A$. 
Theorem

(MD, L. Kurdachenko, A. Pypka, 2014)

Let $G$ be a group and let $\text{Inn } G \leq A \leq \text{Aut } G$. Suppose that $|A/\text{Inn } G| \leq k$ and $|G/C_G(A)| \leq t$. Then $|[G, A]| \leq kt^d$, where $d = \frac{1}{2}(\log p \ t + 1)$, and $p$ is the least prime dividing $t$. 
Results

**Sketch proof** Note that $C_G(A) \leq Z(G)$ so $|G/Z(G)| \leq t$. By Schur, Wiegold theorems $|G'| \leq t^m$ where $m = \frac{1}{2}(\log_p t - 1)$, with $p$ least prime dividing $t$. 
Sketch proof  Note that $C_G(A) \leq Z(G)$ so $|G/Z(G)| \leq t$. By Schur, Wiegold theorems $|G'| \leq t^m$ where $m = \frac{1}{2}(\log_p t - 1)$, with $p$ least prime dividing $t$. If $\alpha \in A$ then $\alpha$ induces $\bar{\alpha} : G_{ab} \rightarrow G_{ab}$. 
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$$[G, A]G'/G' = | \sum_{1 \leq j \leq k} [G_{ab}, \bar{\alpha}_j] | \leq kt.$$
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Then $|[G, A]| \leq tk \cdot t^m$. 

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Then $|[G, A]| \leq tk \cdot t^m$. Is it possible to omit the hypothesis $A/$Inn $G$ finite at the outset?
The Upper $A$-central series

Let $Z_1(G, A) = C_G(A)$ (normal, $A$-invariant). The Upper $A$-central series is

$$1 = Z_0(G, A) \leq Z_1(G, A) \leq Z_2(G, A) \leq \cdots \leq Z_\alpha(G, A) \leq \ldots,$$

where $Z_{\nu+1}(G, A)/Z_\nu(G, A) = Z_1(G/Z_\nu(G, A), A/C_A(Z_\nu(G, A))).$ The usual condition holds for limit ordinals. The last term of this series, is the upper $A$-hypercentre.
The lower $A$-central series of $G$ is the descending, normal $A$-invariant series

$$G = \gamma_1(G, A) \geq \gamma_2(G, A) \geq \cdots \geq \gamma_\nu(G, A) \geq \gamma_{\nu+1}(G, A) \geq \cdots$$

where $\gamma_2(G, A) = [G, A]$ and $\gamma_{\nu+1}(G, A) = [\gamma_\nu(G, A), A]$. Limit ordinals treated as usual. The last term $\gamma_\delta(G, A) = \gamma_\infty(G, A)$ is the lower $A$-hypocentre of $G$. 
Extension of Baer’s Theorem

Theorem

(ND, L. Kurdachenko, A. Pypka, 2014)
Let $G$ be a group and let $A$ be a subgroup of $\text{Aut}(G)$ such that $\text{Inn}(G) \leq A$ and $|A : \text{Inn}(G)| = k$ is finite. Let $Z_\alpha(G, A) = Z$ be the upper $A$-hypercentre of $G$. Suppose that $\alpha = m$ is finite and that $G/Z$ is finite of order $t$. Then

\[|\gamma_{m+1}(G, A)| \leq \beta_k(m, t), \text{ for some function } \beta_k;\]
\[|\gamma_{\infty}(G, A)| \leq \beta_1(k, t), \text{ for some function } \beta_1.\]
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(i) $|\gamma_{m+1}(G, A)| \leq \beta(k, m, t)$, for some function $\beta$;
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Theorem

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Let \( G \) be a group and let \( A \) be a subgroup of \( \text{Aut}(G) \) such that \( \text{Inn}(G) \leq A \) and \( |A : \text{Inn}(G)| = k \) is finite. Let \( Z_\alpha(G, A) = Z \) be the upper \( A \)-hypercentre of \( G \). Suppose that \( \alpha = m \) is finite and that \( G/Z \) is finite of order \( t \). Then

(i) \( |\gamma_{m+1}(G, A)| \leq \beta(k, m, t) \), for some function \( \beta \);

(ii) \( |\gamma_\infty(G, A)| \leq \beta_1(k, t) \), for some function \( \beta_1 \).
Let $G$ be a group and let $A$ be a subgroup of $\text{Aut}(G)$ such that $\text{Inn}(G) \leq A$ and $|A : \text{Inn}(G)| = k$ is finite. If $\gamma_{m+1}(G, A)$ is finite of order $t$ for some positive integer $m$, then $G/\zeta_{2m}(G, A)$ is finite of order at most $\eta(k, m, t)$, for some function $\eta$. 

(MD, L. Kurdachenko, A. Pypka, 2014)
Let $G$ be a group, $F$ a field, $A$ a right $FG$–module. The $FG$-centre of $A$ is

$$Z_{FG}(A) = \{ a \in A \mid a(g - 1) = 0, g \in G \} = C_A(G).$$

Let $\omega FG$ be the augmentation ideal of the group ring $FG$, the two-sided ideal generated by the elements $g - 1$ and let $A(\omega FG)$ be the derived submodule of $A$. 
Linear Case

$G$ has finite section 0-rank $r$ if every torsion-free abelian section has rank at most $r$ and there is such a section of rank $r$. Write $r_0(G) = r$. 

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**Theorem**

*(MD, L. Kurdachenko, J. Otal, 2013)* Let $G \leq GL(F, A)$. Suppose that $\text{codim}_F Z_{FG}(A) \leq c$. If $p$ is 0 or a prime and if $r_p(G) = r < \infty$ then $\text{dim}_F A(\omega FG) \leq \kappa(c, r)$, for some function $\kappa$, where $p$ is the characteristic of $F$. 
Let $A$ have countably infinite dimension over $F$ and let \( \{a_n \mid n \geq 1 \} \) be a basis of $A$. Define $g_k \in GL(F, A)$ by

\[
a_n g_k = \begin{cases} 
a_1 + a_k, & \text{if } n = 1; 
a_n, & \text{if } n > 1.\end{cases}
\]

Write $G = \langle g_k \mid k \in \mathbb{N} \rangle = \text{Dr}_{k \geq 1} \langle g_k \rangle$.

- char $F = 0$ implies $G$ is free abelian.
- char $F = p$ implies $G$ is elementary abelian.

Then $\text{codim}_F Z_{FG}(A) = 1$ but $A(\omega FG)$ is also the subspace generated by $\{a_n \mid n > 1 \}$, so that $A(\omega FG)$ is infinite dimensional.
Let $Z^0_{FG}(A) = 0$, $Z^1_{FG}(A) = Z_{FG}(A)$ and for all ordinals $\alpha$ set $Z^{\alpha+1}_{FG}(A) / Z^\alpha_{FG}(A) = Z_{FG}(A / Z^\alpha_{RG}(A))$, usual convention for limit ordinals. Obtain

$$0 = Z^0_{FG}(A) \leq Z^1_{FG}(A) \leq Z^2_{FG}(A) \leq \cdots \leq Z^\alpha_{FG}(A) \leq \cdots \leq Z^\gamma_{FG}(A)$$

The last term $Z^\gamma_{RG}(A)$ of this series is called the upper $FG$-hypercentre of $A$. Let $A = \gamma^1_{FG}(A)$ and $\gamma^2_{FG}(A) = A(\omega_{FG})$. Let

$$\gamma^{\alpha+1}_{FG}(A) = \gamma^\alpha_{FG}(A)(\omega_{FG})$$

for all ordinals $\alpha$, usual convention for limit ordinals. Obtain

$$A = \gamma^1_{FG}(A) \geq \gamma^2_{FG}(A) \geq \cdots \geq \gamma^\alpha_{FG}(A) \leq \gamma^{\alpha+1}_{FG}(A) \geq \cdots$$
Theorem

(MD, L. Kurdachenko, J. Otal, 2013) Let $G \leq GL(F, A)$. Suppose there exists $k$ such that $\text{codim}_F Z^k_{FG}(A) = c < \infty$. Let $p$ be a prime or 0. If $r_p(G) = r < \infty$ then there exists a function $\lambda$ such that $\text{dim}_F \gamma^{k+1}_{FG}(A) \leq \lambda(c, r, k)$, where $F$ is of characteristic $p$. 
**Theorem**

**(MD, L. Kurdachenko, J. Otal, 2013)** Let $G \leq \text{GL}(F, A)$. Suppose there exists $k$ such that $\text{codim}_F Z_{FG}^k(A) = c < \infty$. Let $p$ be a prime or 0. If $r_p(G) = r < \infty$ then there exists a function $\lambda$ such that $\text{dim}_F \gamma_{FG}^{k+1}(A) \leq \lambda(c, r, k)$, where $F$ is of characteristic $p$.

**Theorem**

**(MD, L. Kurdachenko, J. Otal, 2013)** Let $G \leq \text{GL}(F, A)$. Suppose that there exists $k$ such that $\text{dim}_F \gamma_{FG}^{k+1}(A) = c < \infty$. If $r_p(G) = r < \infty$ there exists a function $\beta$ such that $\text{codim}_F Z_{FG}^{2k}(A) \leq \beta(c, r, k)$. 

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