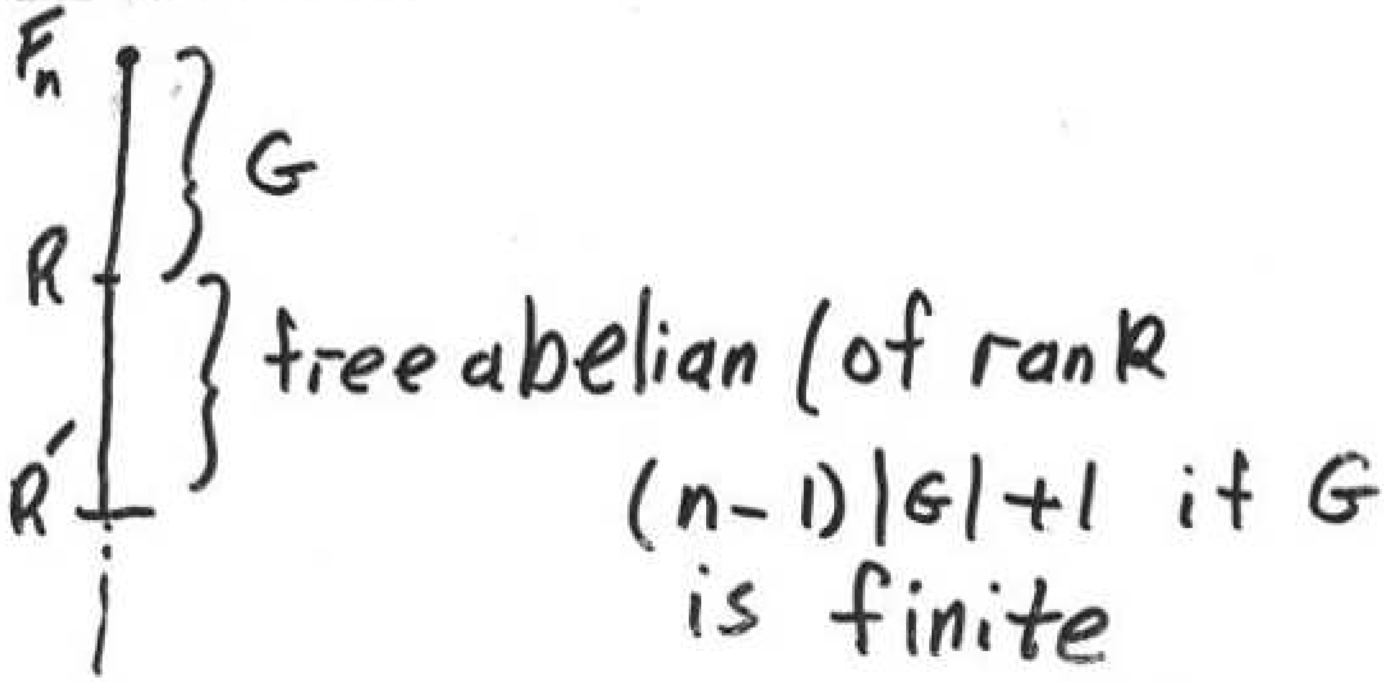
Let G be a finitely generated group, $d(G)$ the minimal number of generators of G, $F_n = \langle x_1, x_2, \dots, x_n \mid \ \rangle$ the free group of rank $n \geq d(G)$ and $R \lhd F_n$ such that $F_n/R \cong G$.

Def$^{\underline{n}}$. F_n/R' is a <u>free</u> <u>abelianized</u> <u>extension</u> of G.



From now on $n>1$.

<u>Ex. 1</u> If $R = F_n^{(m)}$ so that G is free soluble of derived length m and rank n, then $F_n/R' = F_n/F_n^{(m+1)}$ is free soluble of derived length $m+1$ and rank n.

<u>Ex 2</u> If $F_n/R = G$ is the free nilpotent group of class c and rank n, then F_n/R' is the free group of rank n in the variety of all abelian-by-class c groups.

$\underline{\underline{Th^{\underline{m}}\ 1}}$ (L. Auslander and E. Schenkman, 1965)

R/R' is the Hirsch-Plotkin radical of F_n/R'.

(i) R/R' is a characteristic subgroup of F_n/R'

(ii) Every abelian normal subgroup of F_n/R' lies in R/R'

This is all very reasonable.

However, if $R, S \triangleleft F_n$ are such that $F_n/R \cong F_n/S$ it is possible that

$$F_n/R' \not\cong F_n/S'$$

<u>Th$^{\underline{m}}$ 2</u> (M.J. Dunwoody, 1972)

Let $T = \langle x_1, x_2 \mid x_1^2 = x_2^3 \rangle$, the group of the trefoil knot.

There exist $R, S \lhd F_2$ with $F_2/R \cong F_2/S \cong T$ such that

(i) R/R' is the normal closure of a single element in F_2/R'

(ii) S/S' <u>isn't</u>.

Thus $F_2/R' \ncong F_2/S'$.

<u>In fact</u> as $\mathbb{Z}T$-modules

a) $R/R' \cong \mathbb{Z}T$

b) $R/R' \oplus \mathbb{Z}T \cong S/S' \oplus \mathbb{Z}T \cong (\mathbb{Z}T)^2$

c) $S/S' \ncong \mathbb{Z}T$. so S/S' is STABLY FREE BUT NOT FREE.

Recall: A module P over a ring Λ is <u>stably free</u> if $P \oplus \Lambda^r \cong \Lambda^s$ for some r,s.

<u>Th$^{\underline{m}}$ 3</u> If $F_n/R \cong F_n/S \cong G$ is <u>finite</u> and $n > d(G)$ then $F_n/R' \cong F_n/S'$. (P.A. Linnell, 1981)

<u>Th$^{\underline{m}}$</u> 4. There exists a 4-gen group G such that for <u>each</u> integer $n > 4$ there exist $R, S \lhd F_n$, $F_n/R \cong F_n/S \cong G$ but $F_n/R' \not\cong F_n/S'$.

6.

Let $F_n/R = G$, $g_i = x_i R$ and

$M = t_1 \mathbb{Z}G \oplus t_2 \mathbb{Z}G \oplus \cdots \oplus t_n \mathbb{Z}G$ the free $\mathbb{Z}G$-module with basis $t_1, \cdots, t_n$.

Let $\begin{pmatrix} G & 0 \\ M & 1 \end{pmatrix} = \left\{ \begin{pmatrix} g & 0 \\ m & 1 \end{pmatrix} \middle| \begin{matrix} g \in G \\ m \in M \end{matrix} \right\}$,

a group under formal matrix multiplication.

(i) $\begin{pmatrix} g & 0 \\ m & 1 \end{pmatrix}^{-1} = \begin{pmatrix} g^{-1} & 0 \\ -mg^{-1} & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} G & 0 \\ M & 1 \end{pmatrix} \cong G \ltimes M$

$= G \ltimes (t_1 \mathbb{Z}G \oplus \cdots \oplus t_n \mathbb{Z}G)$.

$F_n/R = G$, $g_i = x_i R$.

<u>Magnus</u> <u>embedding</u> <u>F_n/R'</u>

i) $\left\langle \begin{pmatrix} g_1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} g_2 & 0 \\ t_2 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} g_n & 0 \\ t_n & 1 \end{pmatrix} \right\rangle$

$\cong F_n/R'$ (W. Magnus, 1939)

ii) $\begin{pmatrix} g & 0 \\ \sum_{i=1}^{n} t_i \alpha_i & 1 \end{pmatrix}$ lies in this copy of

F_n/R' iff $\sum_{i=1}^{n} (g_i - 1)\alpha_i = g - 1$

(V.N. Remeslenikov and V.G. Sokolo[v]
1970).

The embedding $F_n/R' \hookrightarrow \begin{pmatrix} G & 0 \\ M & 1 \end{pmatrix}$
is the Magnus embedding
or Magnus representation of F_n/R'.

8. Let $A_n = \langle a_1, a_2, \cdots, a_n \rangle$ be the free abelian group of rank n and M_n the free metabelian group of rank n. An element q in a relatively free group V_n of rank n is <u>primitive</u> if $\exists\ q_2, \cdots, q_n \in V_n$ such that $q, q_2, \cdots, q_n$ is a <u>basis</u> of V_n.

<u>Th$^{\underline{m}}$ 5</u> Let q be a primitive element of M_n and suppose that q is contained in the normal closure in M_n of some $y \in M_n$. Then q is conjugate to y or y^{-1}.

<u>Pf</u> View $M_n = \left\langle \begin{pmatrix} a_1 & 0 \\ t_1 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix} \right\rangle$

where $t_1, t_2, \cdots, t_n$ is a basis for $(\mathbb{Z}A_n)^n$.

9.

So M_n consists of all matrices
$$\begin{pmatrix} h & 0 \\ \sum_{i=1}^{n} t_i \alpha_i & 1 \end{pmatrix}, \quad h \in A_n, \ \alpha_i \in \mathbb{Z}A_n.$$
with $\sum_{i=1}^{n} (a_i - 1)\alpha_i = h - 1.$

w.m.a. $y_n = \begin{pmatrix} a_n & 0 \\ t_n & 1 \end{pmatrix}$ is in the normal closure of an element $y = \begin{pmatrix} a_n & 0 \\ t_n + \sum_{i=1}^{n} t_i \alpha_i & 1 \end{pmatrix}$

Step 1 For $i = 1, \cdots, n-1$, $\alpha_i = (a_n - 1)\alpha_i'$

$$y_n^{-1} y = \begin{pmatrix} 1 & 0 \\ \sum t_i \alpha_i & 1 \end{pmatrix} \in y^{M_n} = \langle y, [y, M_n] \rangle$$

so $\begin{pmatrix} 1 & 0 \\ \sum t_i \alpha_i & 1 \end{pmatrix} \in [y, M_n]$

Now $\left[y, \begin{pmatrix} h & 0 \\ m_h & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & \\ (t_n + \sum t_i\alpha_i)(h-1) + m_h(1-a_n) & \end{pmatrix}$

"so" $\sum t_i \alpha_i =$

$$(t_n + \sum t_i \alpha_i)\Big(\sum_{h \in A_n} (h-1) z_h\Big) + \Big(\sum t_i \gamma_i\Big)(1-a_n)$$

"so" for $i = 1, \cdots, n-1$, after equating coefficients of t_i

$$\alpha_i \Big(1 - \sum_{h \in A_n} (h-1) z_h\Big) = \gamma_i (1 - a_n)$$

$z_h \in \mathbb{Z}$, $\gamma_i \in \mathbb{Z}A_n$. Thus

$$\alpha_i \Big(1 - \sum_{h \in A_n} (h-1) z_h\Big) \in (1 - a_n)\mathbb{Z}A_n$$

a <u>prime ideal</u>. Since $1 - \sum (h-1) z_h$ has augmentation 1 we deduce

$$\alpha_i = (1 - a_n)\alpha_i' \text{ as claimed}$$

1.

Step 2 $\alpha_n + 1 = b + \alpha_n'(a_n - 1)$ for some $\alpha_n' \in \mathbb{Z}A_n$, $b \in \pm A_n$.

Uses the fact that the only units in $\mathbb{Z}A_n$ are in $\pm A_n$.

Step 3 finish.

Uses that $\mathbb{Z}A_n$ is a domain.

The Magnus embedding of F_n/R' sometimes allows us to use properties of $\mathbb{Z}(F_n/R)$ to investigate F_n/R' in ways that are explicit and computational.

Fact: Let G be a <u>non-abelian</u> poly-(infinite cyclic) group. Then $\mathbb{Z}G$ has a <u>non-cyclic</u> right ideal I such that $I \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$.
(V.A. Artamonov, 1983; J.T. Stafford, 1985.)

It follows that there exists an epimorphism of $\mathbb{Z}G$-modules
$\phi : t_1\mathbb{Z}G \oplus t_2\mathbb{Z}G \cdots \oplus t_{n+1}\mathbb{Z}G \to t_1\mathbb{Z}G \oplus \cdots \oplus t_n\mathbb{Z}G$
such that $\ker\phi$ is not cyclic. $(n \geq 1)$

Let $G = \mathcal{N}_n = F_n/\gamma_3(F_n)$, the free nilpotent group of class 2 and rank n, and let $g_1, g_2, \ldots, g_n$ be a basis for $\mathcal{N}_n$. Let
$\partial : t_1\mathbb{Z}G \oplus \cdots \oplus t_n\mathbb{Z}G \longrightarrow \mathbb{Z}G$ be
$\partial(t_i) = g_i - 1$.

13. $G = \mathcal{N}_n = F_n/\gamma_3(F_n)$; $g_1, g_2, \cdots, g_n$ a basis of G. $\phi : t_1\mathbb{Z}G \oplus \cdots \oplus t_{n+1}\mathbb{Z}G \twoheadrightarrow t_1\mathbb{Z}G \oplus \cdots \oplus t_n\mathbb{Z}G$ has non-cyclic kernel. $\partial : t_1\mathbb{Z}G \oplus \cdots \oplus t_n\mathbb{Z}G \to \mathbb{Z}G$ is $\partial(t_i) = g_i - 1$.

We can find a map ϕ as above with the additional property $\partial\phi(t_i) = g_i - 1$ for $i = 1, \cdots, n$ and $\partial\phi(t_{n+1}) = 0$. $(n \geq 2)$

<u>Th$^{\underline{m}}$ 6</u> Let U_n be the free abelian-by-(nilpotent of class 2) group of rank n viewed as the Magnus embedding of $F_n/(\gamma_3(F_n)$

i) The homomorphism $\beta : U_{n+1} \to U_n$ given by

$$\begin{pmatrix} g_i & 0 \\ t_i & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} g_i & 0 \\ \phi(t_i) & 1 \end{pmatrix} \quad i = 1, \cdots, n$$

$$\begin{pmatrix} g_{n+1} & 0 \\ t_{n+1} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ \phi(t_{n+1}) & 1 \end{pmatrix}$$

is onto.

$U_n = F_n / (\gamma_3(F_n))'$, $\beta : U_{n+1} \twoheadrightarrow U_n$ $(n \geqslant$

ii) $\ker\beta$ is <u>not</u> the normal closure in U_{n+1} of a single element.

Similar results hold for free soluble groups of derived length $\geqslant 3$ (but not free metabelian groups, C. K. Gupta, N. D. Gupta and G. A. Noskov, 1994).

And now for something completely different, (M. Python, c. 1970).

Let $F_n/R = G$. What can be said about the automorphism group of F_n/R'? What about the group of automorphism that act trivially on F_n/R but not on R/R'?

Recall the Magnus embed of F_n/R' is a subgroup of $\begin{pmatrix} G & \\ (\mathbb{Z}G)^n & 1 \end{pmatrix}$ $\cong G \ltimes (\mathbb{Z}G)^n$. Perhaps we can find automorphisms of $G \ltimes (\mathbb{Z}G)$ hat fix (the embedded) F_n/R'.

16. $F_n/R = G$. Seek automorphisms of F_n/R' that act trivially on F_n/R but not on R/R'.
$F_n/R' \hookrightarrow G \ltimes (\mathbb{Z}G)^n$. Find automorph
of $G \ltimes (\mathbb{Z}G)^n$ that fix F_n/R'.

Let $T : (\mathbb{Z}G)^n \longrightarrow (\mathbb{Z}G)^n$ be a $\mathbb{Z}G$-module automorphism (viewed as an $n \times n$ matrix with entries in $\mathbb{Z}G$) and set $g_i = x_i R$.

If $T = T + W$ where

$$(g_1 - 1, g_2 - 1, \cdots, g_n - 1) W = (0, 0, \cdots, 0)$$

the $F_n/R' \longrightarrow F_n/R'$ given

$$\begin{pmatrix} g_i & 0 \\ t_i & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} g_i & 0 \\ T(t_i) & 1 \end{pmatrix}$$ is

an automorphism of F_n/R'.

 $F_n/R = G$, $g_i = x_i R$. Want $I_n + W \in GL_n(\mathbb{Z}G)$ with $(g_1 - 1, g_2 - 1, \dots, g_n - 1)W = (0, 0, \dots, 0)$.

Th$^{\underline{m}}$ 7 (D.J.S. Robinson, 1982).

Let $F_n/R = G$ be a finite group of order q. Then $\mathrm{Out}(F_n/R')$ has a normal subgroup isomorphic with

$$K(n,q) = \{ X \in GL_n(\mathbb{Z}) \mid X \equiv I_n \bmod q \}$$

Sketch Let $I_n + qA \in K(n,q)$ have inverse $I_n + qB$. Then $I_n = (I_n + qA)(I_n + qB) = I_n + q(A + B + qAB)$ so $A + B + qAB = 0$. Let $\sigma = \sum_{g \in G} g$ and consider $(I_n + \sigma A)(I_n + \sigma B) =$

$I_n + \sigma A + \sigma B + \sigma^2 AB = I_n + \sigma(A + B + qA$

since $\sigma^2 = q\sigma$. Thus $I_n + \sigma A \in GL_n(\mathbb{Z}G$

$\text{Th}^{\underline{m}}$ 8 (V.A. Roman'kov, 1991)

Let $\theta : F_3 \twoheadrightarrow M_3$ be the natural map. Then there exists a primitive element of M_3 that is **not** the image under θ of a primitive element of F_3.

This is a special 'rank 3' result : S. Bachmuth and H. Mochizuki, 1985.

Let's **find** such a 'non-induced' primitive element of M_3.

$A_2 = \langle a_1, a_2 \rangle$, free abelian of rank 2.

$M_2 = \left\langle \begin{pmatrix} a_1 & 0 \\ t_1 & 1 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ t_2 & 1 \end{pmatrix} \right\rangle$, free metab… of rank 2.

It is known that $SL_2(\mathbb{Z}A_2) \neq E_2(\mathbb{Z}A_2$…
(e.g. Bachmuth and Mochizuki, 1982),
a 'rank 2 only' result, (A.A. Suslin, 197…

Th$^{\underline{m}}$ 9 Suppose that

$$A = \begin{pmatrix} 1 + (a_1 - 1)\alpha_1 + (a_2 - 1)\alpha_2 & \beta \\ \gamma & \delta \end{pmatrix}$$

is an element of $SL_2(\mathbb{Z}A_2) \setminus E_2(\mathbb{Z}A$…

Let $\xi_A : F_3 \longrightarrow M_2$ be $x_1 \longrightarrow \begin{pmatrix} a_1 & 0 \\ t_1 + s\alpha_2 & 1 \end{pmatrix}$

$x_2 \longrightarrow \begin{pmatrix} a_2 & 0 \\ t_2 - s\alpha_1 & 1 \end{pmatrix}$, $x_3 \longrightarrow \begin{pmatrix} 1 & 0 \\ s\beta & 1 \end{pmatrix}$

where $s = t_1(a_2 - 1) - t_2(a_1 - 1)$.

20. $A \in SL_2(\mathbb{Z}A_2) \setminus E_2(\mathbb{Z}A_2)$ yields
$\xi_A : F_3 \longrightarrow M_2$

Then ξ is an epimorphism and $\ker \xi$ contains no primitive elements of F_3.

Moreover $p_A = z[z,x]^{\alpha_1}[z,y]^{\alpha_2}[y,x]^{\beta}$ is a non-induced primitive element of $M_3 = \langle x, y, z \rangle$.

Th$^{\underline{m}}$ 10

$$\begin{pmatrix} 1-2(a_1-1)a_2^{-1} & 4a_2^{-1} \\ -(a_1-1)^2 a_2^{-1} & 1+2(a_1-1)a_2^{-1} \end{pmatrix}$$

$\in SL_2(\mathbb{Z}[a_1^{\pm1}, a_2^{\pm1}]) \setminus E_2(\mathbb{Z}[a_1^{\pm1}, a_2^{\pm1}])$

Cor $z[z,y][x,z]^2[y,x]^4$ is a non-induced primitive element of $M_3 = \langle x, y, z \rangle$.