On the Hausdorff spectrum of some groups acting on the p-adic tree

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Measuring the size of subgroups in finite p-groups

All throughout this talk, p denotes a *prime* number.

Let G be a finite p-group, and H a subgroup of G. Then we can use

$$\frac{\log_p |H|}{\log_p |G|}$$

to measure the relative size of H inside G.

Measuring the size of closed subgroups in pro-p groups

Let G be a countably based pro-p group, and $\{G_n\}_{n\in\mathbb{N}}$ a base of neighbourhoods of 1 consisting of open normal subgroups. Then

$$G \cong \varprojlim_{n\to\infty} G/G_n.$$

If H is a *closed* subgroup of G, then

$$H \cong \varprojlim_{n\to\infty} HG_n/G_n.$$

So H can be recovered from its images in the finite p-groups G/G_n.

The relative size of these images is given by

$$\frac{\log_p |HG_n : G_n|}{\log_p |G : G_n|}$$

What if we let $n \to \infty$? The limit need not exist!

Let (X, d) be a metric space, and let $Y \subseteq X$. For real $s \geq 0$ and $\delta > 0$, we define

$$\mathcal{H}^s_\delta(Y) = \inf\Big\{ \sum_{i=1}^{\infty} |U_i|^s \mid \{U_i\} \text{ is a cover of } Y \text{ and } 0 < |U_i| \leq \delta. \Big\}$$

Here $|U_i|$ is the diameter of the set U_i.

Definition

The *s-dimensional Hausdorff measure* of Y is

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(Y).$$

Let (X, d) be a metric space, and let $Y \subseteq X$. Then there exists $s \geq 0$ such that

$$\mathcal{H}^t(Y) = \begin{cases} +\infty, & \text{if } t < s, \\ 0, & \text{if } t > s. \end{cases}$$

Definition

We say that s is the *Hausdorff dimension* of Y, which we write as $\operatorname{Hdim} Y$.

Countably based pro-p groups as metric spaces

Let G be a countably based pro-p group, and $\{G_n\}_{n\in\mathbb{N}}$ a fundamental system of neighbourhoods of 1 consisting of open normal subgroups.

Then

$$d(x,y) = \inf\left\{\frac{1}{|G:G_n|} \mid x \equiv y \pmod{G_n}\right\}$$

is a distance on G.

Measuring the size of closed subgroups in pro-p groups

If G is a countably based pro-p group and H is a closed subgroup of G, the limit

$$\lim_{n\to\infty} \frac{\log_p |HG_n : G_n|}{\log_p |G : G_n|}$$

need not exist, but...

Theorem (Abercrombie, Barnea-Shalev)

With respect to the metric induced by $\{G_n\}_{n\in\mathbb{N}}$, we have

$$\mathrm{Hdim}\, H = \liminf_{n\to\infty} \frac{\log_p |HG_n : G_n|}{\log_p |G : G_n|} \in [0,1].$$

- This value may depend on $\{G_n\}$, the base of neighbourhoods of 1.
- Open subgroups always have Hausdorff dimension 1.

Spectrum of a countably based pro-p group

Definition

If G is a countably based pro-p group then the *spectrum* of G is

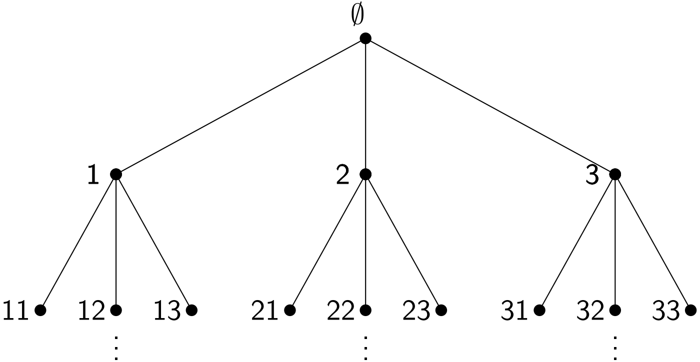
$$\operatorname{Spec} G = \{\operatorname{Hdim} H \mid H \text{ is a closed subgroup of } G\}.$$

Theorem (Barnea-Shalev)

If G is a p-adic analytic pro-p group of dimension d, and we take $G_n = G^{p^n}$ then

$$\operatorname{Spec}(G) \subseteq \left\{0, \frac{1}{d}, \ldots, \frac{d-1}{d}, 1\right\}$$

is finite and consists of rational numbers.



- Here $p = 3$. We denote this tree by $\mathcal{T}_p$, or only $\mathcal{T}$ if p is fixed.
- Vertices are words in the alphabet $\{1, \ldots, p\}$, and $\mathcal{T}$ is structured into *levels* of vertices of the same length.

Automorphisms of the p-adic tree $\mathcal{T}$

An *automorphism* of $\mathcal{T}$ is a bijection of the vertices that preserves incidence. All automorphisms of $\mathcal{T}$ form a group $\operatorname{Aut}\mathcal{T}$.

If $f \in \operatorname{Aut} \mathcal{T}$, then

- f fixes the root $\emptyset$.
- f preserves the levels (f preserves the distance and the n-th level is the sphere of radius n centred at the root).
- The image of a vertex under f determines the images of all its ‘predecessors’.

Automorphisms via labels

Labels of $f \in \operatorname{Aut} \mathcal{T}$ are permutations of S_p which describe, at every vertex v, how f acts on the descendants of v.

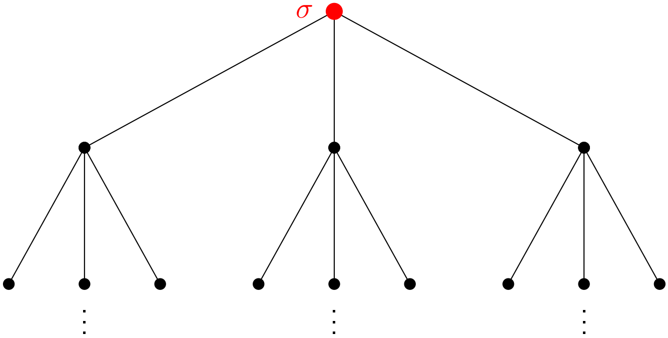
If σ is the label of f at v, then we have

$$f(vi) = f(v)\sigma(i), \quad \text{for every } i = 1, \dots, p.$$

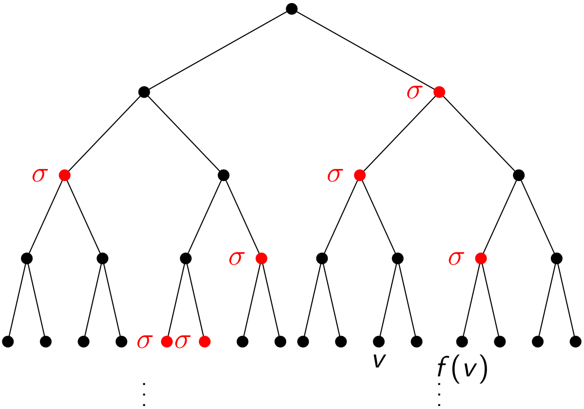
Labels can be used to *describe* a given automorphism, or to *construct* a new one.

A rooted automorphism

What is for example, the automorphism with label $\sigma = (1\ 2\ \ldots\ p)$ at the root, and 1 elsewhere?

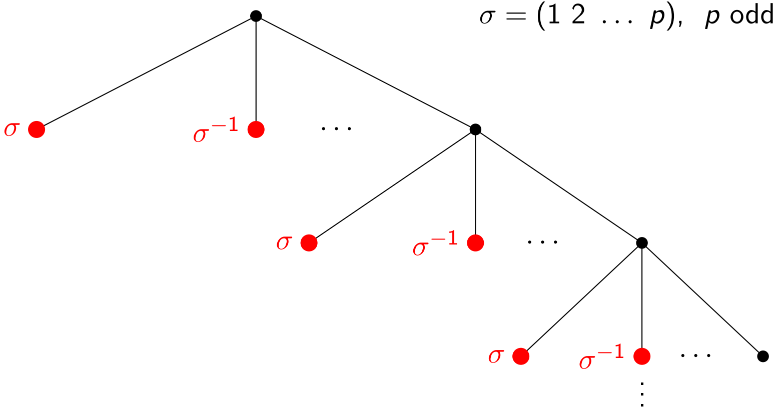


It permutes rigidly the p subtrees under the root as indicated by σ. This is the *rooted automorphism* corresponding to σ.



What is the image of vertex v?

Yet another example...



All labels are 1 except for vertices of the form $p.\overset{i}{\ldots}.p1$ and $p.\overset{i}{\ldots}.p2$ for $i \geq 0$, which are σ and σ^{-1}, respectively.

Let p be odd, and define $a, b \in \operatorname{Aut} \mathcal{T}$ as follows:

- a is the rooted automorphism corresponding to $(1\ 2\ \ldots\ p)$.
- b is the automorphism on the previous slide.

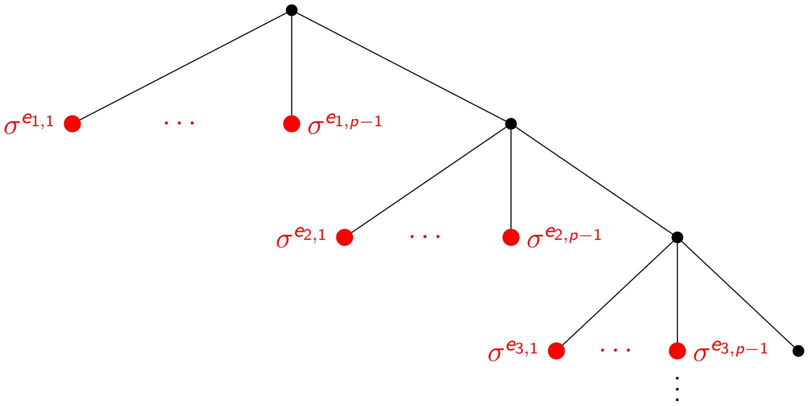
Then both a and b are of order p.

Definition

Then $G = \langle a, b \rangle$ is the *Gupta-Sidki* group for the prime p.

The Gupta-Sidki group is very interesting. For example, it is a counterexample to the General Burnside Problem: it is finitely generated, periodic, but infinite.

Let $\mathbf{E} = (e_{i,j})$ be an integer matrix with infinitely many rows and $p-1$ columns (p odd again). We define $b_{\mathbf{E}}$ via labels:



All labels are 1 except for vertices $p.\overset{i}{.}.pj$, which are $\sigma^{e_{i,j}}$.

Let us write the ith row of $\mathbf{E}$ as

$$\mathbf{e}_i = (e_{i,1}, \ldots, e_{i,p-1}).$$

This is the vector defining labels at the ith level of the tree.

We require that $\mathbf{e}_i \not\equiv (0, \ldots, 0) \pmod{p}$ for all $i \geq 1$, i.e. that no level of the tree has all labels equal to 1.

Then we generalise the Gupta-Sidki group as follows:

$$G = \langle a, b_{\mathbf{E}} \rangle.$$

The ordinary Gupta-Sidki group corresponds to the choice

$$\mathbf{e}_i = (1, -1, 0, \ldots, 0) \text{ for every } i \geq 1.$$

Theorem

If $\mathcal{T}(n)$ denotes the tree $\mathcal{T}$ truncated at the nth level (and including it), then we have

$$\operatorname{Aut}\mathcal{T} \cong \varprojlim_{n\to\infty} \operatorname{Aut}\mathcal{T}(n).$$

So $\operatorname{Aut}\mathcal{T}$ is a profinite group.

Definition

The *nth level stabiliser*, $\operatorname{Stab}(n)$, is the subgroup of $\operatorname{Aut}\mathcal{T}$ consisting of all automorphisms that fix every vertex at the nth level.

Theorem

We have $\operatorname{Aut}\mathcal{T}(n) \cong \operatorname{Aut}\mathcal{T} / \operatorname{Stab}(n)$ and $\{\operatorname{Stab}(n)\}_{n\in\mathbb{N}}$ is a base of neighbourhoods of 1 in $\operatorname{Aut}\mathcal{T}$.

Let us denote by σ the permutation $(1\ 2\ \ldots\ p)$.

Definition

Let Γ be the set of $f \in \operatorname{Aut}\mathcal{T}$ with the property that all labels of f are powers of σ.

Theorem

The group Γ is a pro-p group, and furthermore a Sylow pro-p subgroup of $\operatorname{Aut}\mathcal{T}$. A base of neighbourhoods of 1 is given by the subgroups $\mathrm{Stab}_\Gamma(n) = \Gamma \cap \mathrm{Stab}(n)$.

Observe that the generalised Gupta-Sidki groups $G_{\mathbf{E}}$ introduced before are all subgroups of Γ.

The pro-p group Γ can be seen as a metric space with respect to the distance induced by the subgroups $\{\mathrm{Stab}_\Gamma(n)\}_{n\in\mathbb{N}}$. Recall that

$$d(f,g) = \inf\left\{\frac{1}{|\Gamma : \mathrm{Stab}_\Gamma(n)|} \mid f \equiv g \pmod{\mathrm{Stab}_\Gamma(n)}\right\}.$$

for all $f, g \in \Gamma$.

Theorem (Abért, Virag)

We have $\mathrm{Spec}\,\Gamma = [0,1]$. *More precisely, for every* $s \in [0,1]$ *there exists a* 3*-generator subgroup* G *of* Γ *such that* $\mathrm{Hdim}\,\overline{G} = s$, *where* $\overline{G}$ *is the topological closure of* G.

- However, the proof is probabilistic and no example is given of subgroups of Γ of irrational or transcendental dimension.
- Siegenthaler gave the first examples of transcendental Hausdorff dimension in the particular case $p = 2$.

Hausdorff dimension of generalised Gupta-Sidki groups

Consider first groups $G_{\mathbf{E}}$ where the vectors $\mathbf{e}_i$ are the same for all $i \geq 1$ (for example, the ordinary Gupta-Sidki group).

Let $\mathbf{e} = (e_1, \ldots, e_{p-1})$ be that common vector, and let C be the circulant matrix with first row $(e_1, \ldots, e_{p-1}, 0)$.

Theorem (F-A – Zugadi-Reizabal)

If all $\mathbf{e}_i$ are equal, then

$$\operatorname{Hdim} \overline{G_{\mathbf{E}}} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

where t is the rank of C, and

$$\delta = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1, & \text{if } \mathbf{e} \text{ is constant,} \\ 0, & \text{otherwise.} \end{cases}$$

The proof requires calculating the order of the quotients of $G_{\mathbf{E}}$ by its level stabilisers, an interesting result on its own.

Corollary

If all $\mathbf{e}_i$ are equal, then the Hausdorff dimension is always rational.

But things can be very different if we allow different vectors $\mathbf{e}_i$...

Theorem

By combining symmetric and non-symmetric vectors at different levels in a convenient way, one can find groups of the form $G_{\mathbf{E}}$ with $\operatorname{Hdim} \overline{G_{\mathbf{E}}}$ transcendental.

Note that these are 2-generator groups!

Actually groups of the form $G_{\mathbf{E}}$ give uncountably many different values of the Hausdorff dimension inside Γ.

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