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Some inverse problems in Baumslag-Solitar groups

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Dedicated to the memory of my dear friend and colleague
David Chillag

1946–2012

Introduction

Let G denote an arbitrary group. If X is a **subset** of G, we define:

$$X^2 = \{x_1 x_2 \mid x_1, x_2 \in X\}.$$

Suppose that X is a **finite subset** of G. Since $x_1 X \subseteq X^2$, we have $|X^2| \geq |X|$.
We shall consider problems of the following type:

What if the STRUCTURE of X if $|X^2|$ satisfies

$$|X^2| \leq \alpha|X| + \beta$$

for some small $\alpha \geq 1$ and small $|\beta|$.

Such problems are called INVERSE PROBLEMS of SMALL DOUBLING type.

The first inverse result of "small doubling" type was the following theorem of Gregory Freiman:

Theorem 1. *Let A be a finite subset of a group G*

and suppose that

$$|A^2| < \frac{3}{2}|A|.$$

Then A^2 is a coset of a subgroup of G.

Here the structure of A^2 is determined, not of A. However, if $1 \in A$, then $A \subset A^2$ and it follows that A is a subset of a coset of a subgroup of G of order $|A^2| < \frac{3}{2}|A|$.

This theorem was the beginning of what is now called the "Freiman's structural theory of set addition".

The first version of Theorem 1 was published in 1951 in an obscure Russian journal.

The new version appeared in 2012 in the Proc. Amer. Mat. Soc.,volume **140(9)**.

By now, Freiman's theory had been extended tremendously. It was shown by Freiman and others that problems in various fields may be looked at and treated as Structure Theory problems, including Additive and Combinatorial Number Theory, Group Theory, Integer Programming and Coding Theory.

We noticed recently (see [1], [2] or [3]) that there exists an interesting connection between some *small doubling problems in the Baumslag-Solitar groups* and some results concerning *sums of dilates in the Additive Number Theory*. So we shall dedicate the next two sections to short introductions to Baumslag-Solitar groups and to the theory of sums of dilates.

Baumslag-Solitar groups

The Baumslag-Solitar groups $BS(m, n)$ are two-generated groups with one relation which are defined as follows:

$$BS(m, n) = \langle a, b \mid \; b^{-1}a^m b = a^n \rangle,$$

where m and n are integers. This notation means that the group is the **quotient** of the free group on the two generators a and b by the normal closure of the single element $b^{-1}a^m ba^{-n}$. These groups were introduced by Gilbert Baumslag and Donald Solitar in 1962 in order to provide some simple examples of non-Hopfian groups.

A group is called **Hopfian** (in honor of the topologist Heinz Hopf) if every epimorphism (homomorphism onto) from the group to itself is an isomorphism.

Hence a group is non-Hopfian if it is isomorphic to a proper factor of itself:

$$G \simeq \frac{G}{H} \quad \text{for some} \quad 1 < H \triangleleft G.$$

In 1951, Graham Higman claimed that "every finitely generated group with a single relation is **Hopfian**". In 1954, Bernard Neumann asked:

"Is every two-generator non-Hopfian group **infinitely related**?"

In their 1962 paper Baumslag and Solitar showed that Higman's statement is wrong and the answer to Neumann's question is strongly negative. They showed that the group

$$BS(2,3) = \langle a, b \mid a^{-1}b^2a = b^3 \rangle$$

with **two** generators and **a single relation** is **non-Hopfian**.

More generally, they showed that

$$BS(m,n) \quad \text{is Hopfian}$$

if and only if **one** of the following statements holds:

$$\begin{aligned}(i) & \quad |m| = |n|,\\ (ii) & \quad |m| = 1,\\ (iii) & \quad |n| = 1,\\ (iv) & \quad \pi(m) = \pi(n),\end{aligned}$$

where $\pi(m)$ denotes the set of prime divisors of m.

Clearly $(m,n) = (2,3)$ satisfies none of these conditions, and hence $BS(2,3)$ is **non-Hopfian**.

Sums of dilates

This is a short introduction to the theory of sums of dilates, including two new results which were proved by us.

Let

$$A = \{x_0, x_1, \ldots, x_{k-1}\}$$

be a finite set of integers and let r be positive integer. Define

$$r * A = \{rx_0, rx_1, \ldots, rx_{k-1}\} = \{ra \mid \ a \in A\}.$$

Such sets are called **dilates**. Sums of dilates of the form

$$r * A + s * A = \{ra + sb \mid \ a, b \in A\}$$

have been studied recently by several authors.

For example, the following theorem was proved by Cilleruelo et al. in 2010.

Theorem 2. *If A is a finite set of integers, then*

$$|A + 2 * A| \geq 3|A| - 2$$

and

$$|A + 2 * A| = 3|A| - 2$$

if and only if A is an arithmetic progression.

In the paper [1] we proved the following two new results in this area.
First we showed that for $r \geq 3$, the following stronger result than Theorem 2 holds

Theorem 3. *Let A be a finite set of integers and let r be an integer satisfying $r \geq 3$. Then:*

$$|A + r * A| \geq 4|A| - 4.$$

(Compare to $|A + 2 * A| \geq 3|A| - 2$.).

Our second result deals with an inverse result for $r = 2$:

Theorem 4. *Let A be a finite set of integers. Suppose that $|A| \geq 3$ and*

$$|A + 2 * A| < 4|A| - 4.$$

Then A is a **subset** *of an arithmetic progression of size $\leq 2|A| - 3$.*

This is an example of an inverse result of "sums of dilates" type.

Baumslag-Solitar groups and sums of dilates

We shall describe now the connection between small doubling problems in the Baumslag-Solitar groups $BS(1,n)$ and in sums of dilates. Notice that we are only dealing with the case when $m=1$ and n is an arbitrary non-0 integer.

Let S be a finite subset of $BS(1,n)$ of size k and suppose that S is contained in the coset

$$b^r\langle a\rangle$$

of $\langle a\rangle$ in $BS(1,n)$, for some **positive** integer r. Our assumption that $S\subset b^r\langle a\rangle$ clearly implies that

$$S=\{b^ra^{x_0},b^ra^{x_1},\ldots,b^ra^{x_{k-1}}\},$$

where $A=\{x_0,x_1,\ldots,x_{k-1}\}$ is a set of integers.

We introduce now the notation

$$S = \{b^r a^x : x \in A\} = b^r a^A.$$

Thus $|S| = |A|$. Notice that $\mathbb{Z}$ denotes the set of all integers, while $\mathbb{N}$ denotes the set of **positive** integers.
Since

$$BS(1, n) = \langle a, b \mid \ ab = ba^n \rangle,$$

it follows that $a^{-1}b = ba^{-n}$ and hence

$$a^x b = ba^{nx} \quad \text{for each} \quad x \in \mathbb{Z}.$$

Thus

$$a^x b^r = b^r a^{n^r x}$$

for each $x \in \mathbb{Z}$ and $r \in \mathbb{N}$, yielding

$$(b^r a^x)(b^r a^y) = b^r(a^x b^r)a^y = b^r(b^r a^{n^r x})a^y = b^{2r} a^{n^r x+y}$$

for each $x, y \in \mathbb{Z}$ and for each $r \in \mathbb{N}$. Since

$$S = \{b^r a^x : x \in A\} \quad \text{and} \quad A = \{x_0, x_1, \ldots, x_{k-1}\},$$

it follows that

$$\begin{aligned} S^2 &= \{(b^r a^{x_i})(b^r a^{x_j}) \mid \ i, j \in \{0, 1, \ldots, k-1\}\} \\ &= \{b^{2r} a^{n^r x_i + x_j} \mid \ i, j \in \{0, 1, \ldots, k-1\}\} \\ &= b^{2r} a^{n^r * A + A} \end{aligned}$$

and

$$|S^2| = |n^r * A + A| = |A + n^r * A|.$$

We have proved the following basic result:

Theorem 5. *Suppose that*

$$S = b^r a^A \subseteq BS(1,n)$$

where $r \in \mathbb{N}$ *and* A *is a finite subsets of* $\mathbb{Z}$*. Then*

$$S^2 = b^{2r} a^{n^r * A + A}$$

and

$$|S^2| = |A + n^r * A|.$$

This result served us as the major means for investigating $|S^2|$, using known information about sizes of sums of dilates.

Results in $BS(1,2)$

We now curtail our attention to the Baumslag-Solitar groups $BS(1,2)$. In $BS(1,2)$ the following relation holds:

$$ab = ba^2.$$

Since now $n = 2$, Theorem 5 implies that if $S \subset b\langle a\rangle$, then $|S^2| = |A + 2 * A|$. More generally, if $S \subset b^m\langle a\rangle$ for some positive integer m, then

$$|S^2| = |A + 2^m * A|.$$

Thus Theorems 2,3,4 and 5 yield the following results.

We start with the following proposition

Proposition 6. *Let*

$$S = ba^A \subseteq BS(1,2),$$

where A be a finite set of integers. Then

$$|S^2| \geq 3|S| - 2.$$

Moreover,

$$|S^2| = 3|S| - 2$$

if and only if A **is** *an arithmetic progression.*

Proof. Recall that $|S| = |A|$. By Theorem 5,

$$|S^2| = |A + 2 * A|$$

and by Theorem 2,

$$|A + 2 * A| \geq 3|A| - 2 = 3|S| - 2,$$

yielding

$$|S^2| \geq 3|S| - 2.$$

Moreover,

$$|S^2| = |A + 2 * A| \quad \text{and} \quad 3|S| - 2 = 3|A| - 2.$$

Hence

$$|S^2| = 3|S| - 2 \iff |A + 2 * A| = 3|A| - 2$$

and by Theorem 2, the latter equality holds if and only if A is an arithmetic progression. Hence also $|S^2| = 3|S| - 2$ holds if and only if A is an arithmetic progression, as required.

So if $S = ba^A \subseteq BS(1,2)$, then $|S^2| \geq 3|S| - 2$. But, as a matter of fact, much more is known.

If S is an arbitrary finite nonabelian subset of $BS(1,n)$ for any $n \geq 2$, then

$$|S^2| \geq 3|S| - 2.$$

This follows from the fact that $BS(1,n)$ is an orderable group for $n \geq 2$ and we proved in [4] that if S is an arbitrary finite nonabelian subset of an orderable group, then

$$|S^2| \geq 3|S| - 2.$$

We wish to sketch now the proof of the claim:

If $n \geq 2$, then $BS(1,n)$ is orderable.

Recall the definitions of ordered and orderable groups.

Definition.
Let G be a group and suppose that a total order relation $\leq$ is defined on the set G. We say that $(G, <)$ is an *ordered group* if for all $a, b, x, y \in G$, the inequality $a \leq b$ implies that $xay \leq xby$.

A group G is *orderable*, if there exists a total order relation $\leq$ on the set G, such that $(G, <)$ is an ordered group.

Moreover, by a result of O.Bogopolski (see [5], Lemma 3.1), if $n \geq 2$, then

$$BS(1, n) = U \rtimes V,$$

where

$$U = \langle b^{-j} a b^{j} \mid j \in \mathbb{Z} \rangle \quad \text{and} \quad V = \langle b \rangle.$$

Now by M.I.Kargapolov's condition for orderability of torsion-free groups (see [6]), in order to show that $BS(1, n)$ is orderable, it suffices to show that the following three conditions hold:

(i) $BS(1, n)$ is torsion free for $n \geq 2$;
(ii) The group U is abelian;
(iii) For all $x \in U$, we have $x^b = x^n$.

Concerning (i), it is known that the presentation of $BS(1, n)$ implies that it is torsion-free.

Concerning (ii), it suffices to show that

$$(b^{-i}ab^i)(b^{-j}ab^j) = (b^{-j}ab^j)(b^{-i}ab^i)$$

for all $i, j \in \mathbb{Z}$. We may assume, without loss of generality, that $i > j$ and hence $j - i < 0$. Since $b^{-1}ab = a^n$, it follows that $b^{-r}ab^r \in \langle a \rangle$ for all positive integers r. Hence

$$(b^{j-i}ab^{i-j})a = a(b^{j-i}ab^{i-j}),$$

which implies that

$$b^{-i}ab^{i}b^{-j}ab^{j} = b^{-j}ab^{j}b^{-i}ab^{i}$$

and hence

$$(b^{-i}ab^{i})(b^{-j}ab^{j}) = (b^{-j}ab^{j})(b^{-i}ab^{i}),$$

as required.

Concerning property (iii), it follows from $b^{-1}ab = a^n$ that

$$(b^{-r}ab^{r})^{b} = b^{-r}a^{n}b^{r} = (b^{-r}ab^{r})^{n}$$

for all $r \in \mathbb{Z}$. This implies, by property (ii), that $x^b = x^n$ for all $x \in U$, as required. Hence $BS(1,n)$ is orderable for $n \geq 2$, and by [1] if S is an arbitrary finite nonabelian subset of such $BS(1,n)$, then

$$|S^2| \geq 3|S| - 2.$$

Recall that if $S = ba^A$ is a finite subset of $BS(1,2)$ satisfying

$$|S^2| = 3|S| - 2$$

then by Proposition 6, the structure of S is determined: the set A of integers is an arithmetic progression.

If S is a finite nonabelin subset of an ordered group G, then, as mentioned above, $|S^2| \geq 3|S| - 2$. However, if

$$|S^2| = 3|S| - 2,$$

then the structure of S could be very complicated.

In [7] and [8], both papers under preparation, we succeeded to determine the possible (numerous) structures of subsets S of a torsion free nilpotent group (hence orderable) of class two, satisfying $|S^2| = 3|S| - 2$.

We return now to subsets of $BS(1,2)$. We continue with an inverse result.

Theorem 7. *Let*

$$S = ba^A \subseteq BS(1,2),$$

where A be a finite set of integers and suppose that

$$|S^2| < 4|S| - 4.$$

Then A is a **subset** *of an arithmetic progression of size $\leq 2|A| - 3$.*

This is an **inverse result**, since we know something about $|S^2|$ and we deduce information concerning the **structure** of the set S.

Proof. By Theorems 5 and our assumptions we have

$$|A + 2 * A| = |S^2| < 4|S| - 4 = 4|A| - 4.$$

Hence

$$|A + 2 * A| < 4|A| - 4$$

and Theorem 4 implies that A is a subset of an arithmetic progression of size $\leq 2|A| - 3$.

Our next result deals with sets contained in a coset $b^m\langle a\rangle$ for $m \geq 2$.

Theorem 8. *Let*

$$S = b^m a^A \subseteq BS(1,2),$$

where A be a finite set of integers and $m \geq 2$. Then

$$|S^2| \geq 4|S| - 4.$$

Recall that by Proposition 6, if $ba^A \subseteq BS(1,2)$, then $|S^2| \geq 3|S| - 2$.

Proof. If $r \geq 3$, then by Theorem 3

$$|A + r * A| \geq 4|A| - 4.$$

Since $S = b^m a^A \subseteq BS(1,2)$, Theorem 5 implies that

$$|S^2| = |A + 2^m * A|.$$

As by our assumptions $m \geq 2$, it follows that $2^m > 3$ and hence

$$|S^2| = |A + 2^m * A| \geq 4|A| - 4 = 4|S| - 4,$$

as required.

We notice also the following result concerning

$$S = ba^A \subseteq BS(1, r),$$

where $r \geq 3$.

For such S, Theorems 5 and 3 yield

$$|S^2| = |A + r * A| \geq 4|A| - 4 = 4|S| - 4,$$

while if $S = ba^A \subseteq BS(1,2)$ we get only

$$|S^2| \geq 3|S| - 2.$$

A general inverse result in $BS^+(1,2)$

Until now, we obtained inverse results concerning subsets of $BS(1,2)$, which were contained in **one coset** of $\langle a \rangle$.

We conclude our talk with a general inverse result concerning **arbitrary** finite non-abelian subsets in the monoid $BS^+(1,2)$, to be defined shortly.

In the paper [2], which was submitted for publication, we deal with an inverse problem concerning **arbitrary** finite non-abelian sets S contained in the following subset of $BS(1,2)$:

$$BS^+(1,2) = \{b^m a^x \in BS(1,2)\},$$

where x is an arbitrary integer and m is a non-negative integer. The subset $BS^+(1,2)$ of $BS(1,2)$ is closed with respect to multiplication, so it constitutes a **monoid**.

The advantage of $BS^+(1,2)$ over $BS(1,2)$ is that all elements of $BS^+(1,2)$ can be uniquely represented by a word of the form $b^m a^n$, which is not the case in $BS(1,2)$.

Using rather complicated arguments, we proved the following inverse theorem concerning finite non-abelian subsets S of $BS^+(1,2)$ satisfying the following small doubling condition:

$$|S^2| < \tfrac{7}{2}|S| - 4.$$

Theorem 9. *Let S be a finite non-abelian subset of the monoid $BS^+(1,2)$ and suppose that $|S^2| < \frac{7}{2}|S| - 4$. Then*

$$S = ba^A,$$

where A is a set of integers of size $|S|$, which is contained in an arithmetic progression of size less than $\frac{3}{2}|S|$.

So, under the above condition, the **arbitrary** subset S of $BS^+(1,2)$ is forced to be, after all, a subset of the single coset $b\langle a\rangle$. This result is best possible. In fact, there exist non-abelian subsets S of $BS^+(1,2)$ satisfying $|S^2| = \frac{7}{2}|S| - 4$, which are not contained in one coset of $\langle a\rangle$ in $BS^+(1,2)$.

For example, the set

$$S = a^A \cup \{b\} \subset BS^+(1,2),$$

where

$$A = \{0, 1, 2, \ldots, k-2\} \quad \text{and} \quad k > 2 \quad \text{is even}$$

is non-abelian and satisfies

$$|S^2| = \frac{7}{2}|S| - 4.$$

This set intersects non-trivially two cosets of $\langle a\rangle$ in $BS^+(1,2)$:

$$\langle a\rangle \quad \text{and} \quad b\langle a\rangle.$$

References

[1] *"Direct and Inverse problems in additive number theory and in non-abelian group theory"* by Freiman, Herzog, Longobardi, Maj and Stanchescu, European Journal of Combinatorics, 40(2014), 42-54. This paper contains **all** the topics of this talk, with the exception of the result concerning $BS^+(1,2)$,

[2] *"A small doubling structure theorem in a Baumslag-Solitar group"* same authors, contains the result concerning $BS^+(1,2)$, **submitted** for publication. Papers [1] and [2] (combined) may be found in **[3]** arXiv:1303.3053v1 .

[4] "Small doubling in ordered groups", by Freiman, Herzog, Longobardi and Maj, J. Austra.Mat.Soc. 2014 (in print).

[5] "Abstract commensurators of solvable Baumslag-Solitar groups, "by O.Bogopolski, Communications in Algebra, 40(7), 2012, 2494-2502.

[6] "Completely orderable groups", by M.I.Kargapolov,Algebra i Logica (2) 1 (1962), 16-21.

[7]-[8], Freiman, Herzog, Longobardi, Maj and Stanchescu, under preparation.

The lecture is now complete.

THANK YOU FOR YOUR ATTENTION.