Conjugacy in Relatively Extra-large Artin Groups

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Dedicated to the memory of my friend David Chillag

Definitions

Let n be a natural number, $n \geq 1$, and let $N = (n_{ij})$ be an $n \times n$ symmetric matrix with non-negative integral entries, $n_{ij} \neq 1$ for $1 \leq i, j \leq n$ and $n_{i,i} = 0$, $i = 1, \ldots, n$.

Let F be the free group freely generated by $X := \{x_1, \ldots, x_n\}$.

For $1 \leq i, j \leq n$ such that $n_{ij} \neq 0$ and $n_{ij} \neq 1$ define U_{ij} to be the initial subword of $(x_i x_j)^{n_{ij}}$ of length n_{ij} and define $R_{ij} : U_{ij} U_{ji}^{-1}$.

The Artin group defined by N is the group presented by

$$\langle x_1, \ldots, x_n \mid R_{i,j} = 1\ ,\ 1 \leq i < j \leq n \rangle \quad (1)$$

To each Artin group A there corresponds a *Coxeter group* W_A which is obtained from (1) by adding the relations $x_i^2 = 1$, $i = 1, \ldots, n$.

Examples

1. The Artin group defined by $\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ is given by

$$\langle x_1, x_2, x_3 \mid x_1x_2x_1^{-1}x_2^{-1} = 1 \ , \ x_2x_3x_2^{-1}x_3^{-1} = 1 \ , \ x_1x_3x_1^{-1}x_3^{-1} = 1\rangle$$

and it is the free abelian group on 3 generators.

2. The Artin group defined by $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the free group of rank 3.

3. The Artin group defined by $\begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$ is the braid group B_4 on 4 strands, presented by

$$\langle x_1,\ x_2,\ x_3 \mid x_1x_3 = x_3x_1 \ , \ x_1x_2x_1 = x_2x_1x_2 \ , \ x_2x_3x_2 = x_3x_2x_3\rangle$$

In this talk we concentrate on the conjugacy patterns of elements in a class of Artin groups, primarily on the *Conjugacy Problem* (i.e. the problem of finding an algorithm which for given elements a and b of G decides whether or not a and b are conjugate in A.)

(1.2) **Classes of Artin groups for which the Conjugacy Problem is known to be solvable.**

(i) Finite type, i.e. the corresponding Coxeter groups are finite.

(ii) Right angled, i.e. $n_{ij} \in \{0, 2\}$, for all $1 \leq i, j \leq n$.

(iii) Large and extra-large type, i.e. $n_{ij} \neq 2$, and $n_{ij} \neq 2$ and $n_{ij} \neq 3$, respectively.

(i) For finite type Artin groups the solution is by constructing a normal form, due to Garside, using the fact that these Artin groups are the groups of fractions of the monoid defined by the same presentation. These are the only Artin groups which are groups of fractions of the corresponding monoids.

(ii) For right angled Artin groups the solution heavily relies on the fact that every relation is a commuting relation.

(iii) For large and extra-large Artin groups the solution of the Conjugacy Problem relies on the fact that these groups are small cancellation groups.

We see that the solutions of the Conjugacy Problem in the classes above are very much different from each other and there is no way to find a common generalisation of the solution of the Conjugacy Problem for different classes.

In this work we introduce a new class of Artin groups, which we call *relatively extra-large*, built up from some of the above kinds of Artin groups and for which we provide a solution for the Conjugacy Problem.

(1.3) Relatively extra-large Artin groups - the main results

Recall: a *(standard) parabolic subgroup* H of A is a subgroup of A generated by a subset S of X. Parabolic Artin groups are Artin groups defined by the submatrix of N corresponding to S.

Definition:

Let A be an Artin group and let H be a (standard) parabolic subgroup of A generated by S. A is called *extra-large relative to* H if each of the following holds:

- if $x_i \in S$ and $x_j \notin S$ then $n_{ij} \geq 4$ or $n_{ij} = 0$.
- if $x_i \notin S$ and $x_j \notin S$ then $n_{ij} \geq 3$ or $n_{ij} = 0$.

Thus, if $H = \langle x_1, \ldots, x_k \rangle$, $x_i \in X$, is a parabolic subgroup of A with matrix $N(H)$ then the matrix $N(A)$ of A has the following block decomposition

$N(H)$	$n_{i,j} \notin \{2,3\}$
$n_{i,j} \notin \{2,3\}$	$n_{ij} \neq 2$

Definition

Let G be a 3-generated Artin group. We say that G *satisfies the condition C* if the following holds:

let $Z = x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}$, $|\{i,j,k\}| = 3$, $\alpha_i \neq 0$, $\alpha_i \in \mathbb{Z}$, be an element of $F(x_1, x_2, x_3)$. Then Z is a shortest representative in $F(x_1, x_2, x_3)$ of the element it defines in G.

It is shown (by methods depending on cases) that if G is not of finite type then G satisfies condition C.

It is conjectured that finite-type Artin groups on 3 generators also satisfy condition C.

Definition

Let A be an Artin group with standard generators X. We say that A satisfies *Condition B* if for each $x \in X \cup X^{-1}$ the following condition is satisfied:

There exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that if U and V are shortest words in $F(X)$ which represent elements u and v of A, respectively and $v = y^m u y^{-m}$ for some $y \in X \cup X^{-1}$ and integer $m \neq 0$, then $m \leq f(|U| + |V|)$.

Remark

Each of the classes of Artin groups mentioned above satisfy condition B.

Now we can formulate our main results.

Main Theorem

Let A be an Artin group given by (1) and let H be a parabolic subgroup of A which satisfies both conditions B and C. Suppose that A is extra-large relative to H.

(a) If H has solvable Conjugacy Problem then A has.

(b) If T is a subgroup of H which does not contain x^j, $j \in \mathbb{Z}$, $x \in X$ and T is malnormal in H, then T is malnormal in A.

(c) For every non-cyclic parabolic subgroup P of H we have $N_G(P) = N_H(P)$ and $C_G(P) = C_H(P)$.

Since right angled Artin groups satisfy both conditions B and C and right angled Artin groups have solvable Conjugacy Problem, it follows that Artin groups which are extra-large relative to a right angled parabolic subgroup satisfy results (a), (b) and (c) of the Main Theorem.

(1.4) Sketch of the proof

First, we consider the free product presentation of A, rather than its free presentation given in (1) above:

$$A \cong H * F(T)/\langle\langle \mathcal{R}_1 \rangle\rangle$$

where $X = S \dot{\cup} T$, $H = F(S)/\langle\langle \mathcal{R}_H \rangle\rangle$ and $\mathcal{R} = \mathcal{R}_H \dot{\cup} \mathcal{R}_1$.

This allows us to ignore the lack of knowledge of the properties of H and to consider mainly the part of A outside H and to use its good properties due to the relative extra-largeness. We do this via a detailed analysis of the way H is embedded in A, using annular van Kampen diagrams over the free-product presentation. We show that a certain modification of these diagrams satisfy the small cancellation condition V(6) and use the theory of annular V(6) diagrams to prove a structure theory, which in the case of free presentation would imply the Main Theorem. The matching between the free product presentation and the free presentation is one of the key results. It is achieved by reducing the proof to properties B and C, together with a geometrical property (A-S), which recently has been proven for all Artin groups, by Ruth Charney and Luis Paris.