Capable Special p-Groups of Rank 2

Structure Results

Luise-Charlotte Kappe
Binghamton University
menger@math.binghamton.edu

Joint work with H. Heineken and R.F. Morse

Introduction

Definition

A group G is a special p-group or rank 2 if G has order p^n, $Z(G) = G'$ and $Z(G)$ is an elementary abelian p-group of rank 2.

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A group G is **capable** *if there exists a group H such that $H/Z(H) \cong G$.*

The goal is to determine the capable special p-groups of rank 2 up to isomorphism.

Note: Throughout this talk we assume that p is an odd prime.

Introduction (cont.)

F.R. Beyl, U. Felgner, and P. Schmid, *On groups occurring as center factor groups*, J. of Algebra 61 (1979), 161-177.

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Theorem

An extra-special p-group is capable if and only if it is dihedral of order 8 or order p^3 and exponent p, $p > 2$.

(extra-special = special of rank 1)

Special p-groups of rank 2

H. Heineken, "*Nilpotent groups of class two that can appear as central quotient groups*," Rend. Sem. Mat. Univ. Padova 84 (1991), 241-248.

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Proposition

If G is a finite group such that $C_p \times C_p = G' \subseteq Z(G)$ and there is a group H such that $G \cong H/Z(H)$, then $p^2 < |G/Z(G)| < p^6$.

Special p-groups of rank 2

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Corollary

If G is a special p-group of rank 2 which is capable, then

$$p^5 \leq |G| \leq p^7.$$

Preliminaries

Lemma

Let G be a p-group of nilpotency class 2 whose center is an elementary abelian p-group. Then

(i) $G^p \subseteq Z(G)$;

(ii) G *has exponent at most* p^2;

(iii) $\Phi(G) \subseteq Z(G)$, *where* $\Phi(G)$ *is the Frattini subgroup of* G;

(iv) G *is p-abelian whenever* $p > 2$.

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Split up into two fundamentally different cases:

(1) $\exp(G) = p$;

(2) $\exp(G) = p^2$.

GAP: Isomorphism classes of special p-groups of rank 2, $|G| = p^5$.

	$\exp G = p$		$\exp G = p^2$	
p	Total	Capable	Total	Capable
3	1	1	10	3
5	1	1	12	3
7	1	1	14	3
11	1	1	18	3
13	1	1	20	3
17	1	1	24	3
19	1	1	26	3
23	1	1	30	3
29	1	1	36	3
31	1	1	38	3
37	1	1	44	3

GAP: Isomorphism classes of special p-groups of rank 2, $|G| = p^6$.

	$\exp G = p$		$\exp G = p^2$	
p	Total	Capable	Total	Capable
3	3	3	32	3
5	3	3	38	3
7	3	3	44	3
11	3	3	56	3
13	3	3	62	3
17	3	3	74	3
19	3	3	80	3
23	3	3	92	3
29	3	3	110	3
31	3	3	116	3
37	3	3	134	3

GAP Output: total of isomorphism classes of special p-groups of rank 2 and capable among them for $\exp G = p$ and p^2, $|G| = p^7$.

	$\|G\| = p^7$			
	$\exp G = p$		$\exp G = p^2$	
p	Total	Capable	Total	Capable
3	2	1	97	1
5	2	1	136	1
7	2	1	184	1
11	2	1	298	1

Exponent p

H.A. Bender, "*A determination of groups of order p^5*," Ann. of Math. 29, 61-72, 1927/28.

Rodney James, *The groups of order p^6 (p an odd prime)*," Math. Comp. 34, 613-637, 1980.

David Wilkinson, *The groups of exponent p and order p^7 (p any prime)*," J. Algebra 118, 109-119, 1988.

Proposition: For odd p, the special p-groups of rank 2 and exponent p of order p^5, p^6, and p^7 up to isomorphism are

(1) $\langle x_1, \ldots, x_3, c_1, c_2 \mid [x_2, x_1] = c_1[x_3, x_1] = c_2 \rangle$

(2) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_2, x_1] = c_1, [x_4, x_3] = c_2 \rangle$

(3) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_1, x_2] = c_1, [x_1, x_3] = c_2, [x_2, x_4] = c_2 \rangle$

(4) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_1, x_2] = c_1, [x_1, x_3] = c_2, [x_3, x_4] = c_1, [x_2, x_4] = c_2^g \rangle$ where g is smallest integer root of unity modulo p.

(5) $\langle x_1, \ldots, x_5, c_1, c_2 \mid [x_2, x_1] = [x_5, x_3] = c_1, [x_3, x_1] = [x_5, x_4] = c_2 \rangle$

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Theorem

The groups (1), (2), (3), (4) and (5) are capable.

A. Magidin and R.F. Morse, "*Capable p-groups*," Proceedings Groups St. Andrews 2013.

Exponent p^2

H. Heineken, "*Nilpotent groups of class two that can appear as central quotient groups*," Rend. Sem. Mat. Univ. Padova 84 (1991), 241-248.

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Corollary

If G is a special p-group of rank 2 which is capable, then

$$p^5 \leq |G| \leq p^7.$$

Proposition A

Let G be a p-group of nilpotency class 2. If G^{p^k} is nontrivial and cyclic for some $k \in \mathbb{N}$, then G is not capable, provided that the exponent of G' divides p^k, if p is odd, and the exponent of G' divides p^{k-1}, if $p = 2$.

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Proposition B: Let G be a group with the following presentation:

$$\langle x_1, x_2, y_1, \ldots, y_m \mid x_1^{p^2} = x_2^{p^2} = y_i^p = 1,\ [y_i, y_j] = z_{ij},$$
$$i < j,\ x_1^{x_2} = x_1^{p+1},\ x_1^{y_i} x_1^{s_i p+1} x_2^{t_i p},\ x_2^{y_i} = x_1^{u_i p} x_2^{v_i p+1} \rangle,$$

where p is an odd prime, $z_{ij} \in G^p$, and $0 \leq s_i, t_i, u_i, v_i < p$ for $i = 1, \ldots, m$. Then G has the following properties

(1) G is nilpotent of class 2, has order p^{4+m} and exponent p^2;

(2) $G^p = \langle x_1^p, x_2^p \rangle \cong C_p \times C_p$;

(3) $G' \leq G^p \leq Z(G)$.

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(1) G is nilpotent of class 2, has order p^{4+m} and exponent p^2;
(2) $G^p = \langle x_1^p, x_2^p \rangle \cong C_p \times C_p$;
(3) $G' \le G^p \le Z(G)$.

Observation: The special p-groups of rank 2 and exponent p^2 such that $G^p = C_p \times C_p$ are among the groups represented in Proposition B.

Proposition C

Let G be a group represented in Proposition B with $[y_i, y_j] \neq 1$ for some i, j with $1 \leq i < j \leq m$. Then G is not capable.

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Observation: The special p-groups of rank 2 which are capable are among those groups represented in Proposition B, where $[y_i, y_j] = 1$ for all $1 \leq i < j \leq m$.

Proposition D

Let G be a group represented in Proposition B. Represent the action of y_i on x_1 and x_2 by the matrix

$$m_i = \begin{pmatrix} s_i & t_i \\ u_i & v_i \end{pmatrix}$$

for $i = 1, \ldots, m$. If trace $(m_i) \not\equiv 0 \bmod p$ for some i, $1 \leq i \leq m$, then G is not capable.

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for $i = 1, \ldots, m$. If trace $(m_i) \not\equiv 0$ mod p for some i, $1 \leq i \leq m$, then G is not capable.

Observation: The special p-groups of rank 2 which are capable are among those groups represented in Proposition B, where $s_i + t_i \equiv 0$ mod p for all i and $[y_i, y_j] = 1$ for $1 \leq i < j \leq m$.

$$
\begin{aligned}
G_p(n) = \langle x_1, x_2, y_1, \ldots, y_n \mid & x_1^{p^2} = x_2^{p^2} = y_i^p = [y_i, y_j] = 1, \\
& x_1^{x_2} = x_1^{p+1}, \\
& x_1^{y_i} = x_1^{s_i p+1} x_2^{t_i p}, \\
& x_2^{y_i} = x_1^{u_i p} x_2^{v_i p+1} \rangle
\end{aligned} \tag{1}
$$

where p is an odd prime, $1 \leq i, j \leq n$, $0 \leq s_i, t_i, u_i, v_i < p$, and $(s_i + v_i) \equiv 0 \pmod{p}$.

Theorem 1

Let p be an odd prime and G a special p-group of rank 2 and order p^5 which is capable. Then G has a presentation of the form

$$G = \langle x_1, x_2, y \mid x_1^{p^2} = x_2^{p^2} = y^p = 1,\ x_1^{x_2} = x_1^{p+1},\\ x_1^y = x_1^{sp+1} x_2^{tp},\ x_2^y = x_1^{up} x_2^{vp+1},\\ 0 \leq s, t, u, v < p,\ s + v \equiv 0 \bmod p\rangle.$$

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Theorem 2

Let p be an odd prime and G a special p-group or rank 2 and order p^6 which is capable. Then G has a presentation of the form

$$G = \langle x_1, x_2, y_1, y_2 \mid x_1^{p^2} = x_2^{p^2} = y_1^p = y_2^p = [y_1, y_2] = 1,$$
$$x_1^{x_2} = x_1^{p+1},\ x_1^{y_i} = x_1^{s_i p+1} x_2^{t_i p},\ x_2^{y_i} = x_1^{u_i p} \cdot x_2^{v_i p+1},$$
$$0 \leq s_i, t_i, u_i, v_i < p,\ s_i + v_i \equiv 0 \bmod p,\ i = 1, 2 \rangle.$$

Theorem 3

Let p be an odd prime and G a special p-group of rank 2 and order p^7 which is capable. Then G has a presentation of the form

$$G = \langle x_1, x_2, y_1, y_2, y_3 \mid x_1^{p^2} = x_2^{p^2} = y_1^p = y_2^p = y_3^p = 1,$$
$$[y_i, y_j] = 1,\ 1 \le i < j \le 3,\ x_1^{x_2} = x_1^{p+1},\ x_1^{y_i} = x_1^{s_i p+1} x_2^{t_i p},$$
$$x_2^{y_i} = x_1^{u_i p} x_2^{v_i p+1},\ 0 \le s_i, t_i, u_i, v_i < p,\ s_i + v_i \equiv 0 \bmod p,$$
$$i = 1, 2, 3\rangle.$$