Capable Special $p$-Groups of Rank 2

Structure Results

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Joint work with H. Heineken and R.F. Morse
A group $G$ is a special $p$-group or rank 2 if $G$ has order $p^n$, $Z(G) = G'$ and $Z(G)$ is an elementary abelian $p$-group of rank 2.
Introduction

Definition

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A group $G$ is capable if there exists a group $H$ such that $H/Z(H) \cong G$. 

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Definition
A group $G$ is **capable** if there exists a group $H$ such that $H/Z(H) \cong G$.

The goal is to determine the capable special $p$-groups of rank 2 up to isomorphism.

**Note:** Throughout this talk we assume that $p$ is an odd prime.

**Theorem**

An extra-special $p$-group is capable if and only if it is dihedral of order 8 or order $p^3$ and exponent $p$, $p > 2$.

(extra-special = special of rank 1)
Special $p$-groups of rank 2

Special \( p \)-groups of rank 2


**Proposition**

If \( G \) is a finite group such that \( C_p \times C_p = G' \subset Z(G) \) and there is a group \( H \) such that \( G \cong H/Z(H) \), then \( p^2 < |G/Z(G)| < p^6 \).
Special $p$-groups of rank 2


**Proposition**

If $G$ is a finite group such that $C_p \times C_p = G' \subseteq Z(G)$ and there is a group $H$ such that $G \cong H/Z(H)$, then $p^2 < |G/Z(G)| < p^6$.

**Corollary**

If $G$ is a special $p$-group of rank 2 which is capable, then

$$p^5 \leq |G| \leq p^7.$$
Lemma

Let $G$ be a $p$-group of nilpotency class 2 whose center is an elementary abelian $p$-group. Then

(i) $G^p \subseteq Z(G)$;
(ii) $G$ has exponent at most $p^2$;
(iii) $\Phi(G) \subseteq Z(G)$, where $\Phi(G)$ is the Frattini subgroup of $G$;
(iv) $G$ is $p$-abelian whenever $p > 2$. 
Lemma

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Split up into two fundamentally different cases:

(1) $\exp(G) = p$;
(2) $\exp(G) = p^2$. 
**GAP:** Isomorphism classes of special $p$-groups of rank 2, $|G| = p^5$.

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**GAP:** Isomorphism classes of special $p$-groups of rank 2, $|G| = p^6$.

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**GAP Output:** total of isomorphism classes of special \( p \)-groups of rank 2 and capable among them for \( \exp G = p \) and \( p^2 \), \(|G| = p^7\).

| \( p \) | \(|G| = p^7\) | \( \exp G = p \) | Total | Capable | \( \exp G = p^2 \) | Total | Capable |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 1 | 97 | 1 |
| 5 | 2 | 1 | 136 | 1 |
| 7 | 2 | 1 | 184 | 1 |
| 11 | 2 | 1 | 298 | 1 |
Exponent $p$


Proposition: For odd $p$, the special $p$-groups of rank 2 and exponent $p$ of order $p^5$, $p^6$, and $p^7$ up to isomorphism are

(1) $\langle x_1, \ldots, x_3, c_1, c_2 \mid [x_2, x_1] = c_1, [x_3, x_1] = c_2 \rangle$

(2) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_2, x_1] = c_1, [x_4, x_3] = c_2 \rangle$

(3) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_1, x_2] = c_1, [x_1, x_3] = c_2, [x_2, x_4] = c_2 \rangle$

(4) $\langle x_1, \ldots, x_4, c_1, c_2 \mid [x_1, x_2] = c_1, [x_1, x_3] = c_2, [x_3, x_4] = c_1, [x_2, x_4] = c_2^g \rangle$ where $g$ is smallest integer root of unity modulo $p$.

(5) $\langle x_1, \ldots, x_5, c_1, c_2 \mid [x_2, x_1] = [x_5, x_3] = c_1, [x_3, x_1] = [x_5, x_4] = c_2 \rangle$

(6) $\langle x_1, \ldots, x_5, c_1, c_2 \mid [x_2, x_1] = c_1, [x_4, x_3] = c_2, [x_5, x_4] = c_1 \rangle$
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**Theorem**

The groups (1), (2), (3), (4) and (5) are capable.


Proposition

If $G$ is a finite group such that $C_p \times C_p = G' \subseteq Z(G)$ and there is a group $H$ such that $G \cong H/Z(H)$, then $p^2 < |G/Z(G)| < p^6$.

Corollary

If $G$ is a special $p$-group of rank 2 which is capable, then

$$p^5 \leq |G| \leq p^7.$$
Proposition A

Let $G$ be a $p$-group of nilpotency class 2. If $G^{p^k}$ is nontrivial and cyclic for some $k \in \mathbb{N}$, then $G$ is not capable, provided that the exponent of $G'$ divides $p^k$, if $p$ is odd, and the exponent of $G'$ divides $p^{k-1}$, if $p = 2$. 

Proposition B:

Let $G$ be a group with the following presentation:

\[
\langle x_1, x_2, y_1, \ldots, y_m | x_1^{p^2} = x_2^{p^2} = y_i^{p^i} = 1, [y_i, y_j] = z_{ij}, i < j, x_1 x_2 = x_1^{p+1}, x_1 y_1 x_2 s_i x_1^{p+1} x_1 t_i, x_1 y_2 x_2 u_i x_1^{p} x_1^{p+1}, \rangle
\]

where $p$ is an odd prime, $z_{ij} \in G^{p^i}$, and $0 \leq s_i, t_i, u_i, v_i < p$ for $i = 1, \ldots, m$. Then $G$ has the following properties:

1. $G$ is nilpotent of class 2, has order $p^4 + m$ and exponent $p^2$;
2. $G^{p^2} = \langle x_1^{p^2}, x_2^{p^2} \rangle \cong C_p \times C_p$;
3. $G' \leq G^{p^2} \leq Z(G)$.

Observation: The special $p$-groups of rank 2 and exponent $p^2$ such that $G^{p^2} = C_p \times C_p$ are among the groups represented in Proposition B.
Proposition A

Let $G$ be a $p$-group of nilpotency class 2. If $G^{p^k}$ is nontrivial and cyclic for some $k \in \mathbb{N}$, then $G$ is not capable, provided that the exponent of $G'$ divides $p^k$, if $p$ is odd, and the exponent of $G'$ divides $p^{k-1}$, if $p = 2$.

Proposition B: Let $G$ be a group with the following presentation:

$$\langle x_1, x_2, y_1, \ldots, y_m \mid x_1^{p^2} = x_2^{p^2} = y_i^p = 1, \ [y_i, y_j] = z_{ij}, \ i < j, \ x_1^{x_2} = x_1^{p+1}, \ x_1^{y_i} x_1^{s_i p+1} x_2^{t_i p}, \ x_2^{y_i} = x_1^{u_i p} x_2^{v_i p+1} \rangle,$$

where $p$ is an odd prime, $z_{ij} \in G^p$, and $0 \leq s_i, t_i, u_i, v_i < p$ for $i = 1, \ldots, m$. Then $G$ has the following properties

(1) $G$ is nilpotent of class 2, has order $p^{4+m}$ and exponent $p^2$;
(2) $G^p = \langle x_1^p, x_2^p \rangle \cong C_p \times C_p$;
(3) $G' \leq G^p \leq Z(G)$.
Proposition A

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Proposition B: Let $G$ be a group with the following presentation:

$$\langle x_1, x_2, y_1, \ldots, y_m \mid x_1^{p^2} = x_2^{p^2} = y_i^p = 1, [y_i, y_j] = z_{ij},
\text{ for } i < j, x_1^{x_2} = x_1^{p+1}, x_1^{y_i} x_1^{s_i p+1} x_2^{t_i p}, x_1^{y_i} = x_1^{u_i p} x_2^{v_i p+1} \rangle,$$

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Observation: The special $p$-groups of rank 2 and exponent $p^2$ such that $G^p = C_p \times C_p$ are among the groups represented in Proposition B.
Proposition C

Let $G$ be a group represented in Proposition B with $[y_i, y_j] \neq 1$ for some $i, j$ with $1 \leq i < j \leq m$. Then $G$ is not capable.
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Observation: The special $p$-groups of rank 2 which are capable are among those groups represented in Proposition B, where $[y_i, y_j] = 1$ for all $1 \leq i < j \leq m$. 
Let $G$ be a group represented in Proposition B. Represent the action of $y_i$ on $x_1$ and $x_2$ by the matrix

$$m_i = \begin{pmatrix} s_i & t_i \\ u_i & v_i \end{pmatrix}$$

for $i = 1, \ldots, m$. If $\text{trace}(m_i) \not\equiv 0 \pmod{p}$ for some $i$, $1 \leq i \leq m$, then $G$ is not capable.
Proposition D

Let $G$ be a group represented in Proposition B. Represent the action of $y_i$ on $x_1$ and $x_2$ by the matrix

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Observation: The special $p$-groups of rank 2 which are capable are among those groups represented in Proposition B, where $s_i + t_i \equiv 0 \pmod{p}$ for all $i$ and $[y_i, y_j] = 1$ for $1 \leq i < j \leq m$. 
\[ G_p(n) = \langle x_1, x_2, y_1, \ldots, y_n \mid x_1^{p^2} = x_2^{p^2} = y_i^p = [y_i, y_j] = 1, \quad (1) \]
\[ x_1^{x_2} = x_1^{p+1}, \]
\[ x_1^{y_i} = x_1^{s_i p+1} x_2^{t_i p}, \]
\[ x_2^{y_i} = x_1^{u_i p} x_2^{v_i p+1} \]

where \( p \) is an odd prime, \( 1 \leq i, j \leq n, 0 \leq s_i, t_i, u_i, v_i < p \), and \((s_i + v_i) \equiv 0 \pmod{p}\).
Theorem 1

Let $p$ be an odd prime and $G$ a special $p$-group of rank 2 and order $p^5$ which is capable. Then $G$ has a presentation of the form

$$G = \langle x_1, x_2, y \mid x_1^{p^2} = x_2^{p^2} = y^p = 1, x_1^{x_2} = x_1^{p+1},$$

$$x_1^y = x_1^{sp+1} x_2^{tp}, \quad x_2^y = x_1^{up} x_2^{vp+1},$$

$$0 \leq s, t, u, v < p, \quad s + v \equiv 0 \mod p \rangle.$$
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$$x_1^y = x_1^{sp+1} x_2^{tp}, \ x_2^y = x_1^{up} x_2^{vp+1},$$
$$0 \leq s, t, u, v < p, \ s + v \equiv 0 \mod p \rangle.$$
Theorem 3

Let $p$ be an odd prime and $G$ a special $p$-group of rank 2 and order $p^7$ which is capable. Then $G$ has a presentation of the form

$$G = \langle x_1, x_2, y_1, y_2, y_3 \mid x_1^{p^2} = x_2^{p^2} = y_1^p = y_2^p = y_3^p = 1,$$
$$[y_i, y_j] = 1, \ 1 \leq i < j \leq 3, \ x_1^{y_2} = x_1^{p+1}, \ x_1^{y_i} = x_1^{s_ip+1}x_2^{t_ip},$$
$$x_2^{y_i} = x_1^{u_ip}x_2^{v_ip+1}, \ 0 \leq s_i, t_i, u_i, v_i < p, \ s_i + v_i \equiv 0 \mod p, \ i = 1, 2, 3 \rangle.$$