Finite groups admitting Frobenius groups of automorphisms with almost fixed-point-free kernel

Evgeny Khukhro

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Frobenius group

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Frobenius groups often occur in finite groups;
induce groups of automorphisms by conjugation

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The condition $C_G(F) = 1$ alone (with F being nilpotent) already implies many nice properties of G:

G is soluble (Belyaev–Hartley + CFSG),

the Fitting height of G is bounded in terms of $\alpha(|F|)$ (Thompson–. . .),

and if $|F|$ is a prime, then G is nilpotent of class $\leqslant h(|F|)$ (Higman–Krekinin–Kostrikin).

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Roughly speaking, "$V = |H|$ times $C_V(H)$"

(as $C_V(H) =$ diagonal).

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Suppose that a finite group G admits a Frobenius group of automorphisms $FH \leqslant \mathrm{Aut}\, G$ with kernel F and complement H such that $C_G(F) = 1$.

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— the order of $|G|$;

— the rank of G;

— the Fitting height of G; by means of representation theory

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— if G is nilpotent, for the nilpotency class by a Lie ring method (true only if FH is metacyclic);

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So far, only known for FH metacyclic (EKh–N. Makarenko–P. Shumyatsky) and $FH \cong A_4$ (P. Shumyatsky).

New results

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— relaxing the strong conditions on the action of the kernel or/and

— relaxing the conditions on the structure of the group FH itself.

Almost fixed-point-free kernel

An important next step is considering finite groups G with a Frobenius group of automorphisms FH with 'almost fixed-point-free' kernel F:

expecting properties of G to be 'almost as good' match to those of $C_G(H)$ as in the case of fixed-point-free kernel F'.

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Theorem (EKh, 2013)

Suppose that a finite p-group P admits a Frobenius group of automorphisms FH with kernel F and complement H such that the orders of P and FH are coprime: $(|P|, |FH|) = 1$. *If* $P = [P, F]$, *then*

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(c) *the rank of P is bounded above in terms of $|H|$ and the rank of $C_P(H)$.*

Order and rank: any finite group of coprime order

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(b) $\mathbf{r}(G) \leqslant \mathbf{r}(C_G(F)) + g(|H|, \mathbf{r}(C_G(H)))$ *for some function g of two variables.*

Nilpotency

Theorem (EKh–N. Makarenko, 2013)

Suppose that a finite group G admits a Frobenius group of automorphisms FH of coprime order with kernel F and complement H such that $C_G(H)$ is nilpotent. Then $|G : F(G)|$ is bounded in terms of $|C_G(F)|$ and $|F|$.

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Then the proof is by representation theory (Clifford's theorem, ...).

Nilpotency class

Theorem (EKh–N. Makarenko, 2013)

Suppose that a finite group G admits a metacyclic Frobenius group of automorphisms FH of coprime order such that $C_G(H)$ is nilpotent of class c. Then G has a nilpotent subgroup of index bounded in terms of c, $|C_G(F)|$, and $|F|$ whose nilpotency class is bounded in terms of c and $|H|$ only.

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Then a Lie ring method of "graded centralizers" is applied, similarly to results on almost fixed-point-free automorphism of prime order. The previous EKh–Makarenko–Shumyatsky nilpotency theorem for fixed-point-free kernel is used in place of Higman–Kreknin–Kostrikin theorem

Unipotent kernel: nilpotency class

Theorem (EKh–N. Makarenko, 2013)

Suppose that a finite p-group P admits a metacyclic Frobenius group FH of automorphisms with kernel F of order p^k, and let c be the nilpotency class of $C_P(H)$.

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Examples show that such a result no longer holds if the "metacyclic" condition on FH is dropped.

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Proof is based on a Lie ring method.
The previous EKh–Makarenko–Shumyatsky nilpotency theorem for fixed-point-free kernel is used in a combinatorial form.

Unipotent kernel: rank, order, exponent

Theorem (EKh–N. Makarenko, 2013)

Suppose that a finite p-group P admits a Frobenius group FH of automorphisms with cyclic kernel F of order p^k. Then P has a characteristic subgroup Q of index bounded in terms of $|F|$ and $|C_P(F)|$ such that

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Here, it is not clear what to do for non-cyclic F.

Frobenius-like groups of automorphisms

Further progress: Relaxing the conditions on FH itself:

Frobenius-like groups: semidirect product FH with nilpotent F such that $FH/[F,F]$ is a Frobenius group with kernel $F/[F,F]$ and complement H.

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for Frobenius-like groups of automorphisms $FH \leqslant \mathrm{Aut}\, G$,
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mainly on bounding the Fitting height of G
in terms of the Fitting heigth of $C_G(H)$.

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Theorem (Ercan–Güloğlu, 2013)

Suppose that a finite group G admits a Frobenius-like group of automorphisms FH of coprime order such that the kernel F is extraspecial, $[Z(F), H] = 1$, and FH satisfies certain arithmetical conditions (e.g. is of odd order). Then the Fitting height of G is equal to the Fitting height of $C_G(H)$.

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Similar result was also obtained for the case H of prime order by Collins (student of Flavell).