Homogenous Finitary Symmetric Groups

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$$FSym(\Omega) = \{\alpha \in Sym(\Omega) : \; |Supp(\alpha)| < \infty\}.$$

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An easy example of an infinite locally finite, simple group of cardinality κ is $Alt(\Omega)$ where $|\Omega| = \kappa$.

For $\Omega = \mathbb{N}$ we have

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$$S_2 \leq S_3 \leq S_4 \leq \ldots$$

and

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If we take the non-identity embedding of S_n into $S_{k_i n}$ and then into $S_{k_{i+1} k_i n}$ where $(k_1, k_2, \ldots, k_i, \ldots)$ be a given infinite sequence of positive integers and we take the embeddings with respect to the given sequence of infinitely many integers and we take the union of these groups, then we have a new infinite locally finite group.

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What are the structure of such groups?
This is a natural method to produce infinite, simple locally finite groups.

Construction of Groups of type $S(\xi)$

Definition. The embedding d of the transitive permutation group (G, X) into the permutation group (H, Y) is called a **diagonal embedding** if the restriction of $d(G)$ on every orbit of length greater than one is permutation isomorphic to (G, X).

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A diagonal embedding is called a **strictly diagonal embedding** if the length of every orbit of the image $d(G)$ on the set Y is greater than one.

Let α be the permutation defined by

$$\alpha = \begin{pmatrix} 1 \ldots & n \\ i_1 \ldots & i_n \end{pmatrix}$$

Then the permutation

$$d^r(\alpha) = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \ldots & n & n+1 & \ldots & 2n & \ldots & (r-1)n+1 & \ldots & rn \\ i_1 & \ldots & i_n & n+i_1 & \ldots & n+i_n & \ldots & (r-1)n+i_1 & \ldots & (r-1)n+i_n \end{array} \right)$$

is called a **homogeneous r-spreading** of the permutation α.

Example. Consider the permutation

$$\alpha = (12) \in Sym(3).$$

Embed α into $Sym(9)$ by homogenous 3-spreading.

We obtain

$$d^3(\alpha) = \left(\begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 5 & 6 & 6 & 8 & 7 & 9 \end{array}\right) = (1,2)(4,5)(7,8)$$

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Let $\xi \in \Pi$ and $\xi = (p_1, p_2, \ldots)$ be a sequence consisting of not necessarily distinct primes p_i.
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$$\{1\} \stackrel{d^{p_1}}{\longrightarrow} S_{n_1} \stackrel{d^{p_2}}{\longrightarrow} S_{n_2} \stackrel{d^{p_3}}{\longrightarrow} S_{n_3} \stackrel{d^{p_4}}{\longrightarrow} \ldots$$

$$\{1\} \stackrel{d^{p_1}}{\longrightarrow} A_{n_1} \stackrel{d^{p_2}}{\longrightarrow} A_{n_2} \stackrel{d^{p_3}}{\longrightarrow} A_{n_3} \stackrel{d^{p_4}}{\longrightarrow} \ldots$$

where $n_i = n_{i-1}p_i, \quad i = 1, 2, 3\ldots$ and S_{n_i} is the symmetric group on n_i letters and A_{n_i} is the alternating group on n_i letters and $n_0 = 1$.

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The direct limit groups are denoted by $S(\xi)$ and $A(\xi)$ respectively.

The construction of these groups are discussed in

Kegel-Wehfritz's book, Locally Finite Groups [2], (1976)

and then studied by
N. V. Kroshko-V. I. Sushchansky in [4],

Direct limits of symmetric and alternating groups with strictly diagonal embeddings, Arch. Math. 71, (1998).

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- If the prime 2 appears infinitely often in the sequence ξ, then the direct limit group $S(\xi)$ is a simple non-linear, non-finitary locally finite group.
- If a prime p appears infinitely often, then $S(\xi)$ contains an isomorphic copy of the locally cyclic p-group C_{p^∞}.
- The group $FSym(\mathbb{N})$ does not contain C_{p^∞} for any prime p.

Recall that the formal product $n = 2^{r_2}3^{r_3}5^{r_5}\dots$ of prime powers with $0 \leq r_k \leq \infty$ for all k is called a **Steinitz number** (supernatural number).

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The set of Steinitz numbers form a partially ordered set with respect to division, namely if $\alpha = 2^{r_2}3^{r_3}5^{r_5}\dots$ and $\beta = 2^{s_2}3^{s_3}5^{s_5}\dots$ be two Steinitz numbers, then $\alpha|\beta$ if and only if $r_p \leq s_p$ for all prime p.

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Moreover they form a lattice if we define meet and join as $\alpha \wedge \beta = 2^{min\{r_2,s_2\}}3^{min\{r_3,s_3\}}5^{min\{r_5,s_5\}}\dots$ and $\alpha \vee \beta = 2^{max\{r_2,s_2\}}3^{max\{r_3,s_3\}}5^{max\{r_5,s_5\}}\dots$.

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For each Steinitz number ξ we can define a strictly diagonal group $S(\xi)$ and for each strictly diagonal group $S(\xi)$ we have a Steinitz number.

The set of groups $S(\xi)$ ordered with respect to being subgroup form a lattice.

For each sequence ξ we define $Char(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}} \ldots$ where r_{p_i} is the number of times that the prime p_i repeat in ξ. If it repeats infinitely often, then we write p_i^{∞}.

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Theorem 3 *(Kuroshko-Sushchansky, 1998)*
Two groups $S(\xi_1)$ and $S(\xi_2)$ are isomorphic if and only if $Char(S(\xi_1)) = Char(S(\xi_2))$.

The groups $S(\xi)$ are classified by Kroshko-Sushchansky by using the lattice of Steinitz numbers.

There are uncountably many non-isomorphic simple locally finite groups of type $S(\xi)$.

Moreover Kroshko-Sushchansky proved that there is a lattice isomorphism between the lattice of groups $S(\xi)$ and the lattice of Steinitz numbers.

Centralizers in groups $S(\xi)$.

Since our groups $S(\xi)$ are obtained as direct limits of finite symmetric groups we remind you centralizers of elements in finite symmetric groups.

Let $g \in S_n$. The **cycle type** of g is denoted by $t(g) = (r_1, r_2, \ldots r_n)$ where r_i is the number of cycles of length i in the cycle decomposition of g.

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Example. For $g = (12)(34)\ldots(n-1n)$ with k, 2-cycles where $2k = n$, then the centralizer is

$$C_{S_n}(g) \ \cong \ C_2 \wr S_k$$

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For an element $g \in S(\xi)$ we define the principal beginning as g_0 where $g_0 \in S_{n_i}$ and n_i is the smallest in the chain (changing the order is allowed to take the minimum).

Theorem 4 *(Güven, Kegel, Kuzucuoğlu [1]) Let* ξ *be an infinite sequence,* $g \in S(\xi)$ *and the type of principal beginning* $g_0 \in S_{n_k}$ *be* $t(g_0) = (r_1, r_2, \ldots, r_{n_k})$. *Then*

$$C_{S(\xi)}(g) \cong \operatorname*{Dr}_{i=1}^{n_k} C_i(C_i \bar{\wr}\, S(\xi_i))$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_k} r_i \quad$ *for* $i = 1, \ldots, n_k$.
If $r_i = 0$, *then we assume that corresponding factor is* $\{1\}$.

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Example. For $Char(\xi) = 2357\ldots$ and $\alpha_0 = (12)$ we have

$$C_{S(\xi)}(\alpha) \cong \langle\alpha\rangle(C_2 \bar{\wr} S(\xi_1)\ )$$

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where $Char(S(\xi_1)) = \frac{Char(S(\xi))}{2} = 357\ldots.$.

Example. Let $Char(\xi) = 5^{\infty}$ and $g_0 = (12345)$. Then $C_{S(\xi)}(g) \cong \langle g\rangle(C_5 \bar{\wr} S(\xi))$

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Theorem 6 *(Güven, Kegel, Kuzucuoğlu [1]) Let F be a finite subgroup of the infinite group $S(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of F such that the action of F on any two orbits in Γ_i are equivalent. Let the type of F be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then*

$$C_{S(\xi)}(F) \cong \underset{i=1}{\overset{k}{Dr}} \, (C_{Sym(\Omega_i)}(F|_{\Omega_i})(C_{Sym(\Omega_i)}(F|_{\Omega_i})\bar{\wr}\; S(\xi_i)))$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_{j_i}} r_i$ and Ω_i is a representative of an orbit in the equivalence class Γ_i for $i = 1, \ldots, k$.

Now we construct homogenous finitary symmetric groups $FSym(\kappa)(\xi)$.

Let κ be an arbitrary infinite cardinal number.
Let $FSym(\kappa)$ denote the finitary symmetric group and $Alt(\kappa)$ denote the alternating group on the set κ.

As before, let Π be the set of sequences of prime numbers and $\xi \in \Pi$. Then ξ is a sequence of not necessarily distinct primes.

Let $\alpha \in FSym(\kappa), \quad (\ Alt(\kappa)\ )$. For a natural number $p \in \mathbb{N}$ a permutation $d^p(\alpha) \in FSym(\kappa p)$ defined by $(\kappa s + i)^{d^p(\alpha)} = \kappa s + i^\alpha, \quad i \in \kappa$ and $0 \leq s \leq p-1$ is called **homogeneous** p-**spreading** of the permutation α.

We divide the ordinal κp to p equal parts and on each part we repeat the permutation diagonally as in the finite case. So if $\alpha = \begin{pmatrix} 1 \dots & n \\ \vdots & \vdots \\ i_1 \dots & i_n \end{pmatrix} \in FSym(\kappa)$, then the homogeneous $p-$spreading of the permutation α is

$$d^p(\alpha) = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \dots & n & \kappa+1 & \dots & \kappa+n & \dots & \kappa(p-1)+1 & \dots & \kappa(p-1)+n \\ i_1 & \dots & i_n & \kappa+i_1 & \dots & \kappa+i_n & \dots & \kappa(p-1)+i_1 & \dots & \kappa(p-1)+i_n \end{array}\right)$$

with the assumption that the elements in $\kappa p \setminus supp(d^p(\alpha))$ are fixed.

We continue to take the embeddings using homogeneous p-spreadings with respect to the given sequence of primes in ξ.

From the given sequence of embeddings, we have direct systems and hence direct limit groups $FSym(\kappa)(\xi)$, $(Alt(\kappa)(\xi))$.

Observe that $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are subgroups of $Sym(\kappa\omega)$.

Lemma 7 *If the characteristics of two homogenous finitary symmetric groups are different (homogenous infinite alternating groups), then the groups are non-isomorphic.*

By the above there exists uncountably many pairwise non-isomorphic simple locally finite groups for each cardinality κ.

Theorem 8 *(Kegel, Kuzucuoğlu, [3]) Let* κ *be an infinite cardinal. If* $G = \bigcup_{i=1}^{\infty} G_i$, *where* $G_i = FSym(\kappa n_i)$, *is a group of strictly diagonal type and* $\xi = (p_1, p_2, \ldots)$, *then* G *is isomorphic to the homogenous finitary symmetric group* $FSym(\kappa)(\xi)$, *where* $n_0 = 1, \quad n_i = p_1 p_2 \ldots p_i, \quad i \in \mathbb{N}$.

The principal beginning α_0 of an element $\alpha \in FSym(\kappa)(\xi)$ is defined to be the smallest positive integer $n_j \in \mathbb{N}$ such that $\alpha_0 \in FSym(\kappa n_j)$ and α_0 is not obtained as a sequence of embeddings d^{p_i} for any $p_i \in \xi$.

Theorem 9 *(Güven, Kegel, Kuzucuoğlu [1]) Let* ξ *be an infinite sequence. If* $\alpha \in FSym(\kappa)(\xi)$ *with principal beginning* $\alpha_0 \in FSym(\kappa n_i), \quad t(\alpha_0) = (r_1, \ldots, r_n)$, *and* $|supp(\alpha_0)| = n$.

Then

$$C_{FSym(\kappa)(\xi)}(\alpha) \cong (\overset{n}{\underset{i=1}{Dr}}\; C_i(C_i \bar{\wr} S(\xi_i))) \times FSym(\kappa)(\xi')$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_i} r_i$ *and* $Char(\xi') = \frac{Char(\xi)}{n_i}$.
If $r_i = 0$, *then we assume that the corresponding factor in the direct product is* $\{1\}$.

Theorem 10 *Let κ be a fixed infinite cardinal. There is a lattice isomorphism between the lattice of groups* $\Sigma = \{FSym(\kappa)(\xi) \mid \xi \in \Pi\ \}$ *ordered with respect to being a subgroup and the lattice $\mathcal{S}$ of Steinitz numbers ordered with respect to division in Steinitz numbers.*

Theorem 11 *Let ξ be an infinite sequence of not necessarily distinct primes. Let F be a finite subgroup of $FSym(\kappa)(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of F such that the action of F on any two orbits in Γ_i is equivalent. Let the type of F be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then*

$$C_{FSym(\kappa)(\xi)}(F) \cong (\operatorname*{Dr}_{i=1}^{k} C_{Sym(\Omega_i)}(F)(C_{Sym(\Omega_i)}(F)\bar{\wr}\, S(\xi_i)) \times FSym(\kappa)$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_{j_i}} r_i$ and $Char(\xi') = \frac{Char(\xi)}{n_{j_1}}$ and Ω_i is a representative of an orbit in the equivalence class Γ_i for $i = 1, \ldots, k$.

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THANK YOU