Homogenous Finitary Symmetric Groups

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Then

$$F\text{Sym}(\Omega) = \{ \alpha \in \text{Sym}(\Omega) : |\text{Supp}(\alpha)| < \infty \}.$$
The finitary symmetric group $FSym(\Omega)$ is a locally finite group of cardinality equal to the cardinality of $\Omega$. 
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Recall that a group is called a **locally finite group** if every finitely generated subgroup is a finite group.

An easy example of an infinite locally finite, simple group of cardinality $\kappa$ is $\text{Alt}(\Omega)$ where $|\Omega| = \kappa$.

For $\Omega = \mathbb{N}$ we have

$$F\text{Sym}(\mathbb{N}) = \bigcup_{n=1}^{\infty} \text{Sym}(n)$$
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$$S_2 \leq S_3 \leq S_4 \leq \ldots$$

and

$$FSym(\mathbb{N}) = \bigcup_{n=1}^{\infty} S_n$$
If we take the non-identity embedding of $S_n$ into $S_{k_i n}$ and then into $S_{k_i+1, k_i n}$ where $(k_1, k_2, \ldots, k_i, \ldots)$ be a given infinite sequence of positive integers and we take the embeddings with respect to the given sequence of infinitely many integers and we take the union of these groups, then we have a new infinite locally finite group.
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What are the structure of such groups? This is a natural method to produce infinite, simple locally finite groups.
Construction of Groups of type $S(\xi)$
**Definition.** The embedding $d$ of the transitive permutation group $(G, X)$ into the permutation group $(H, Y)$ is called a **diagonal embedding** if the restriction of $d(G)$ on every orbit of length greater than one is permutation isomorphic to $(G, X)$. 
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A diagonal embedding is called a **strictly diagonal embedding** if the length of every orbit of the image \(d(G)\) on the set \(Y\) is greater than one.
Let \( \alpha \) be the permutation defined by

\[
\alpha = \begin{pmatrix} 1 \ldots n \\ i_1 \ldots i_n \end{pmatrix}
\]

Then the permutation

\[
d^r(\alpha) = \\
\begin{pmatrix}
1 \ldots n \\
i_1 \ldots i_n
\end{pmatrix}
\begin{pmatrix}
n + 1 \ldots 2n \\
n + i_1 \ldots n + i_n
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\vdots
\end{pmatrix}
\begin{pmatrix}
(r - 1)n + 1 \ldots rn \\
(r - 1)n + i_1 \ldots (r - 1)n + i_n
\end{pmatrix}
\]

is called a \textbf{homogeneous} \( r \)-spreading of the permutation \( \alpha \).
Example. Consider the permutation

\[ \alpha = (12) \in \text{Sym}(3). \]

Embed \( \alpha \) into \( \text{Sym}(9) \) by homogenous 3-spreading.

We obtain

\[ d^3(\alpha) = \left( \begin{array}{ccc|ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 3 & 5 & 6 & 6 & 8 & 7 & 9 \end{array} \right) = (1, 2)(4, 5)(7, 8) \]
Let \( \Pi \) be the set of sequences consisting of prime numbers. Let \( \xi \in \Pi \) and \( \xi = (p_1, p_2, \ldots) \) be a sequence consisting of not necessarily distinct primes \( p_i \).
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$$\{1\} \xrightarrow{d^{p_1}} S_{n_1} \xrightarrow{d^{p_2}} S_{n_2} \xrightarrow{d^{p_3}} S_{n_3} \xrightarrow{d^{p_4}} \ldots$$

$$\{1\} \xrightarrow{d^{p_1}} A_{n_1} \xrightarrow{d^{p_2}} A_{n_2} \xrightarrow{d^{p_3}} A_{n_3} \xrightarrow{d^{p_4}} \ldots$$

where $n_i = n_{i-1} p_i$, $i = 1, 2, 3 \ldots$ and $S_{n_i}$ is the symmetric group on $n_i$ letters and $A_{n_i}$ is the alternating group on $n_i$ letters and $n_0 = 1$. 
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The direct limit groups are denoted by $S(\xi)$ and $A(\xi)$ respectively.
The construction of these groups are discussed in Kegel-Wehfritz’s book, Locally Finite Groups [2], (1976) and then studied by N. V. Kroshko-V. I. Sushchansky in [4],

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• If the prime 2 appears infinitely often in the sequence $\xi$, then the direct limit group $S(\xi)$ is a simple non-linear, non-finitary locally finite group.

• If a prime $p$ appears infinitely often, then $S(\xi)$ contains an isomorphic copy of the locally cyclic $p$-group $C_p\infty$.

• The group $FSym(\mathbb{N})$ does not contain $C_p\infty$ for any prime $p$. 

Recall that the formal product \( n = 2^{r_2}3^{r_3}5^{r_5} \ldots \) of prime powers with \( 0 \leq r_k \leq \infty \) for all \( k \) is called a Steinitz number (supernatural number).
Recall that the formal product $n = 2^{r_2} 3^{r_3} 5^{r_5} \ldots$ of prime powers with $0 \leq r_k \leq \infty$ for all $k$ is called a **Steinitz number** (supernatural number).

The set of Steinitz numbers form a partially ordered set with respect to division, namely if $\alpha = 2^{r_2} 3^{r_3} 5^{r_5} \ldots$ and $\beta = 2^{s_2} 3^{s_3} 5^{s_5} \ldots$ be two Steinitz numbers, then $\alpha | \beta$ if and only if $r_p \leq s_p$ for all prime $p$. 
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Moreover they form a lattice if we define meet and join as
\[
\alpha \land \beta = 2^{\min\{r_2,s_2\}} 3^{\min\{r_3,s_3\}} 5^{\min\{r_5,s_5\}} \ldots \quad \text{and} \\
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\]

For each Steinitz number \( \xi \) we can define a strictly diagonal group \( S(\xi) \) and for each strictly diagonal group \( S(\xi) \) we have a Steinitz number.
The set of groups $S(\xi)$ ordered with respect to being subgroup form a lattice.
For each sequence $\xi$ we define $Char(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}} \ldots$ where $r_{p_i}$ is the number of times that the prime $p_i$ repeat in $\xi$. If it repeats infinitely often, then we write $p_i^\infty$. 
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Therefore for each $\xi \in \Pi$, there corresponds a Steinitz number $Char(\xi)$. For a group $S(\xi)$ obtained from the sequence $\xi$ we define $Char(S(\xi)) = Char(\xi)$.
For each sequence $\xi$ we define $\text{Char}(\xi) = p_1^{r_{p_1}} p_2^{r_{p_2}} \ldots$ where $r_{p_i}$ is the number of times that the prime $p_i$ repeat in $\xi$. If it repeats infinitely often, then we write $p_i^\infty$.

Therefore for each $\xi \in \Pi$, there corresponds a Steinitz number $\text{Char}(\xi)$. For a group $S(\xi)$ obtained from the sequence $\xi$ we define $\text{Char}(S(\xi)) = \text{Char}(\xi)$.

**Theorem 3** (Kuroshko-Sushchansky, 1998) Two groups $S(\xi_1)$ and $S(\xi_2)$ are isomorphic if and only if $\text{Char}(S(\xi_1)) = \text{Char}(S(\xi_2))$. 
The groups $S(\xi)$ are classified by Kroshko-Sushchansky by using the lattice of Steinitz numbers.

There are uncountably many non-isomorphic simple locally finite groups of type $S(\xi)$.

Moreover Kroshko-Sushchansky proved that there is a lattice isomorphism between the lattice of groups $S(\xi)$ and the lattice of Steinitz numbers.
Centralizers in groups $S(\xi)$. 
Since our groups $S(\xi)$ are obtained as direct limits of finite symmetric groups we remind you centralizers of elements in finite symmetric groups.
Let $g \in S_n$. The **cycle type** of $g$ is denoted by $t(g) = (r_1, r_2, \ldots r_n)$ where $r_i$ is the number of cycles of length $i$ in the cycle decomposition of $g$. 
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Then

\[
C_{S_n}(g) \cong \bigotimes_{i=1}^{n} C_i \wr S_{r_i}
\]

if \( r_i = 0 \), then we assume the corresponding factor is \( \{1\} \).
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**Example.** For $g = (12)(34)\ldots(n - 1 n)$ with $k$, 2-cycles where $2k = n$, then the centralizer is

$$C_{S_n}(g) \cong C_2 \wr S_k$$
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For an element $g \in S(\xi)$ we define the principal beginning as $g_0$ where $g_0 \in S_{n_i}$ and $n_i$ is the smallest in the chain (changing the order is allowed to take the minimum).
Theorem 4 (Güven, Kegel, Kuzucuoğlu [1]) Let \( \xi \) be an infinite sequence, \( g \in S(\xi) \) and the type of principal beginning \( g_0 \in S_{n_k} \) be \( t(g_0) = (r_1, r_2, \ldots, r_{n_k}) \). Then

\[
C_S(\xi)(g) \cong \bigotimes_{i=1}^{n_k} C_i(C_i \wr S(\xi_i))
\]

where \( \text{Char}(\xi_i) = \frac{\text{Char}(\xi)}{n_k} r_i \) for \( i = 1, \ldots, n_k \).

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For the centralizers of elements we have the following examples:
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**Example.** For $Char(\xi) = 2357 \ldots$ and $\alpha_0 = (12)$ we have

$$C_{S(\xi)}(\alpha) \cong \langle \alpha \rangle (C_{\bar{2}l}S(\xi_1))$$

where $Char(S(\xi_1)) = \frac{Char(S(\xi))}{2} = 357 \ldots$. 
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$$C_{S(\xi)}(\alpha) \cong \langle \alpha \rangle (C_2 \wr S(\xi_1))$$

where $Char(S(\xi_1)) = \frac{Char(S(\xi))}{2} = 357\ldots$.

**Example.** Let $Char(\xi) = 5^\infty$ and $g_0 = (12345)$. Then

$$C_{S(\xi)}(g) \cong \langle g \rangle (C_5 \wr S(\xi))$$
For the centralizers of finite subgroups in $S(\xi)$, we have the following Theorem.
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**Theorem 6** (Güven, Kegel, Kuzucuoğlu [1]) Let $F$ be a finite subgroup of the infinite group $S(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of $F$ such that the action of $F$ on any two orbits in $\Gamma_i$ are equivalent. Let the type of $F$ be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then

$$C_{S(\xi)}(F) \cong \prod_{i=1}^{k} \left( C_{Sym(\Omega_i)}(F|\Omega_i)(C_{Sym(\Omega_i)}(F|\Omega_i)^\perp S(\xi_i)) \right)$$

where $Char(\xi_i) = \frac{\text{Char}(\xi)}{n_{j_i}} r_i$ and $\Omega_i$ is a representative of an orbit in the equivalence class $\Gamma_i$ for $i = 1, \ldots, k$. 
Now we construct homogenous finitary symmetric groups $FSym(\kappa)(\xi)$.
Let $\kappa$ be an arbitrary infinite cardinal number. Let $FSym(\kappa)$ denote the finitary symmetric group and $Alt(\kappa)$ denote the alternating group on the set $\kappa$.

As before, let $\Pi$ be the set of sequences of prime numbers and $\xi \in \Pi$. Then $\xi$ is a sequence of not necessarily distinct primes.

Let $\alpha \in FSym(\kappa)$, $(Alt(\kappa))$. For a natural number $p \in \mathbb{N}$ a permutation $d^p(\alpha) \in FSym(\kappa p)$ defined by

$$(\kappa s + i)^{d^p(\alpha)} = \kappa s + i^\alpha, \quad i \in \kappa \text{ and } 0 \leq s \leq p - 1$$

is called homogeneous $p$-spreading of the permutation $\alpha$. 


We divide the ordinal \( \kappa p \) to \( p \) equal parts and on each part we repeat the permutation diagonally as in the finite case. So if 
\[
\alpha = \begin{pmatrix} 1 \ldots n \\ i_1 \ldots i_n \end{pmatrix} \in FSym(\kappa),
\]
then the homogeneous \( p \)-spreading of the permutation \( \alpha \) is

\[
d^p(\alpha) = \begin{pmatrix}
1 & \ldots & n & \kappa + 1 & \ldots & \kappa + n & \ldots & \kappa(p - 1) + 1 & \ldots & \kappa(p - 1) + n \\
i_1 & \ldots & i_n & \kappa + i_1 & \ldots & \kappa + i_n & \ldots & \kappa(p - 1) + i_1 & \ldots & \kappa(p - 1) + i_n
\end{pmatrix}
\]

with the assumption that the elements in \( \kappa p \setminus supp(d^p(\alpha)) \) are fixed.
We continue to take the embeddings using homogeneous $p$-spreadings with respect to the given sequence of primes in $\xi$.

From the given sequence of embeddings, we have direct systems and hence direct limit groups $FSym(\kappa)(\xi)$, $(Alt(\kappa)(\xi))$.

Observe that $FSym(\kappa)(\xi)$ and $Alt(\kappa)(\xi)$ are subgroups of $Sym(\kappa\omega)$. 
Lemma 7  If the characteristics of two homogenous finitary symmetric groups are different (homogenous infinite alternating groups), then the groups are non-isomorphic.

By the above there exists uncountably many pairwise non-isomorphic simple locally finite groups for each cardinality $\kappa$. 
Theorem 8  (Kegel, Kuzucuoğlu, [3]) Let \( \kappa \) be an infinite cardinal. If \( G = \bigcup_{i=1}^{\infty} G_i \), where \( G_i = FSym(\kappa n_i) \), is a group of strictly diagonal type and \( \xi = (p_1, p_2, \ldots) \), then \( G \) is isomorphic to the homogenous finitary symmetric group \( FSym(\kappa)(\xi) \), where \( n_0 = 1, \; n_i = p_1 p_2 \ldots p_i, \; i \in \mathbb{N} \).
The principal beginning $\alpha_0$ of an element $\alpha \in FSym(\kappa)(\xi)$ is defined to be the smallest positive integer $n_j \in \mathbb{N}$ such that $\alpha_0 \in FSym(\kappa n_j)$ and $\alpha_0$ is not obtained as a sequence of embeddings $d^{p_i}$ for any $p_i \in \xi$. 
Theorem 9 (Güven, Kegel, Kuzucuoğlu [1]) Let $\xi$ be an infinite sequence. If $\alpha \in F Sym(\kappa)(\xi)$ with principal beginning $\alpha_0 \in F Sym(\kappa n_i)$, $t(\alpha_0) = (r_1, \ldots, r_n)$, and $|supp(\alpha_0)| = n$. Then

$$C_{F Sym(\kappa)(\xi)}(\alpha) \cong \left( \prod_{i=1}^{n} F Sym(\kappa n_i) \right) \times F Sym(\kappa)(\xi')$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_i} r_i$ and $Char(\xi') = \frac{Char(\xi)}{n_i}$. If $r_i = 0$, then we assume that the corresponding factor in the direct product is $\{1\}$. 
Theorem 10  Let \( \kappa \) be a fixed infinite cardinal. There is a lattice isomorphism between the lattice of groups
\[ \Sigma = \{ F \text{Sym}(\kappa)(\xi) \mid \xi \in \Pi \} \]
ordered with respect to being a subgroup and the lattice \( S \) of Steinitz numbers ordered with respect to division in Steinitz numbers.
**Theorem 11**  Let $\xi$ be an infinite sequence of not necessarily distinct primes. Let $F$ be a finite subgroup of $FSym(\kappa)(\xi)$ and $\Gamma_1, \ldots, \Gamma_k$ be the set of orbits of $F$ such that the action of $F$ on any two orbits in $\Gamma_i$ is equivalent. Let the type of $F$ be $t(F) = ((n_{j_1}, r_1), (n_{j_2}, r_2), \ldots, (n_{j_k}, r_k))$. Then

$$C_{FSym(\kappa)(\xi)}(F) \cong \prod_{i=1}^{k} C_{Sym(\Omega_i)}(F) \tilde{\wr} S(\xi_i) \times FSym(\kappa)$$

where $Char(\xi_i) = \frac{Char(\xi)}{n_{ji}} r_i$ and $Char(\xi') = \frac{Char(\xi)}{n_{j_1}}$

and $\Omega_i$ is a representative of an orbit in the equivalence class $\Gamma_i$ for $i = 1, \ldots, k$. 
References


THANK YOU