Another presentation for symplectic Steinberg groups

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April 5, 2014

Ischia Group Theory

Picture

R — any commutative unital ring, $n \geq 3$

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$\mathrm{Sp}(2n, R)$

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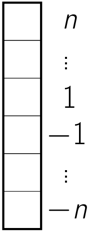

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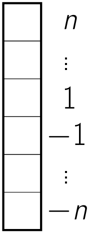
$T_{ij}(a): \ w \mapsto w + e_i a w_j - e_{-j} a w_{-i} \varepsilon_i \varepsilon_j, \ \ \varepsilon_k = \mathrm{sign}(k)$



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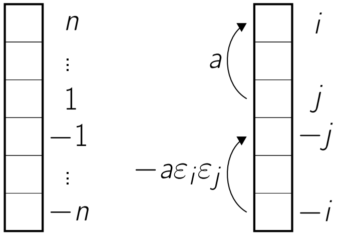

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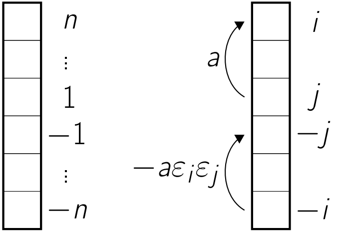

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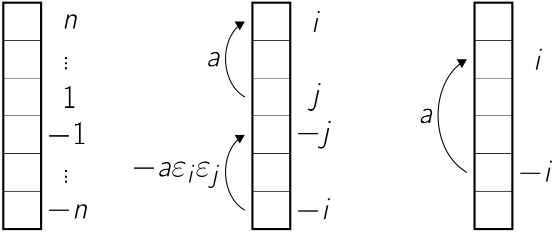

Definition

$$\mathrm{StSp}(2n,\ R) = \langle \{X_{ij}(a),\ \ i \neq j,\ \ a \in R\} \mid$$

$$
\begin{aligned}
&X_{ij}(a) = X_{-j,-i}(-a\varepsilon_i\varepsilon_j), && \text{(R0)}\\
&X_{ij}(a)X_{ij}(b) = X_{ij}(a+b), && \text{(R1)}\\
&[X_{ij}(a),\ X_{hk}(b)] = 1,\ \text{ for } h \notin \{j,-i\},\ \ k \notin \{i,-j\}, && \text{(R2)}\\
&[X_{ij}(a),\ X_{jk}(b)] = X_{ik}(ab),\ \text{ for } i \notin \{-j,-k\},\ j \neq -k, && \text{(R3)}\\
&[X_{i,-i}(a),\ X_{-i,j}(b)] = X_{ij}(ab\varepsilon_i)X_{-j,j}(-ab^2), && \text{(R4)}\\
&[X_{ij}(a),\ X_{j,-i}(b)] = X_{i,-i}(2ab\varepsilon_i)\rangle && \text{(R5)}
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$$\begin{array}{cccc}
\phi : & \mathrm{StSp}(2n,\ R) & \longrightarrow & \mathrm{Sp}(2n,\ R)\\
& \Cup & & \Cup\\
& X_{ij}(a) & \mapsto & T_{ij}(a)
\end{array}$$

Proof

Theorem (Suslin–Kopeiko–Taddei)

$\mathrm{Im}(\phi) \trianglelefteq \mathrm{Sp}(2n, R), \ \ n \geq 2$

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$$T(u, v, a) : \ w \mapsto w + u\Big(\underbrace{(v, w)}+a\underbrace{(u, w)}\Big) + v\underbrace{(u, w)}_{\text{symplectic form on } R^{2n}}$$

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$$T_{ij}(a) = T(e_i, e_{-j}a\varepsilon_{-j}, 0), \qquad T_{i,-i}(a) = T(e_i, 0, a)$$

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$\mathrm{Ker}(\phi) \subseteq \mathrm{Cent}\,\mathrm{StSp}(2n,\ R),\ \ n \geq 3$

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How to define $X(u,\ v,\ a) \in \mathrm{StSp}(2n,\ R)$ such that
$g\,X(u,\ v,\ a)g^{-1} = X(\phi(g)u,\ \phi(g)v,\ a) \quad \forall g \in \mathrm{StSp}(2n,\ R)$?

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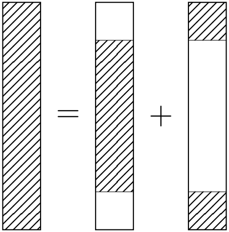
W. van der Kallen, Another presentation for Steinberg groups, 1976

New result



$$X(u, 0, a) = X(\tilde{u}, 0, a)X(u', 0, a)X(u', \tilde{u}a, 0)$$

New result

Corollary

$$\mathrm{H}_2(\mathrm{Im}(\phi),\, \mathbb{Z}) \rightarrowtail \mathrm{Ker}(\phi) \twoheadrightarrow \mathrm{H}_2(\mathrm{StSp}(2l,\, R), \mathbb{Z})$$

In particular, this map is bijective for $l \geq 4$ or for $l = 3$ and R having no residue field isomorphic to $\mathbb{F}_2$.

Corollary

$$\mathrm{K}_2\mathrm{Sp}^{Q}(2l,\, R) \rightarrowtail \mathrm{Ker}(\phi) \twoheadrightarrow \mathrm{H}_2(\mathrm{StSp}(2l,\, R), \mathbb{Z})$$

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