And thanks for the outstanding hospitality

A QUESTION FOR ISCHIA

Arising from joint work with Bettina Eick, Mike Newman, and Eamonn O'Brien, and many others, too many to mention, including three other speakers at this conference.

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Is it possible to partition the class of all p-groups into sub-classes in such a way that

(i) If P and Q are p-groups of the same order and nilpotency class then P and Q belong to the same subclass.

(ii) Each subclass contains infinitely many isomorphism classes.

(iii) It is possible to determine efficiently (polynomial time?) for each subclass, how many isomorphism clases of groups of order p^n lie in the given subclass?

Slow answer: Perhaps.

A partially discredited attempt

Suppose, for example, that we take the sub-classes to be indexed by the natural numbers n, and take the n-th class to contain all p-groups of order p^n, as p varies over all primes p.

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Does this work?

Graham Higman had a conjecture that states:
The number of isomorphism types of groups of order p^n, for fixed n, is PORC as a function of p. That is to say, is Polynomial On Residue Classes.
Suppose that someone produces a constructive proof. That is to say, it is proved that:
For every $n > 0$, there exist integers s and k, and s polynomials f_0, $f_1, \ldots, f_{s-1}$, all of degree at most k, such that the number of isomorphisms classes of p-groups of order p^n is $f_i(p)$, where $p \equiv i \bmod s$. Moreover s and k are at most $\cdots$ and $\cdots$ respectively.

But suppose the conjecture holds

Would such a theorem, the given bounds making the proof 'explicit', solve the problem?

Suppose that you ask for the number of isomorphism classes of groups of order 1789^{24}, and that 1789 is prime.

The idea is that you give me first the exponent 24, and then, when I am ready, you give me the prime 1789, and then I give you an instantaneous answer.

So given the exponent 24 I compute the number of isomorphism groups of order p^{24} for $p = 2, 3, 5, 7, 11, \ldots$, and when I have computed enough of these values I can deduce the integer s and the polynomials f_i.

Now, when you give me the prime 1789, I determine $i = 1789 \bmod s$, and evaluate $f_i(1789)$.

Not a practical classification

First objection.
In practice, one cannot start this calculation.

Second objection.
In any case, the set of primes p for which one would have to calculate the number of isomorphism classes of groups of order p^{24} would probably exceed 1789, so the algoritm would in any case be useless unless for primes such as 37861937832977, if this is a prime.

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Reply to the objections.
An explicit positive solution to the PORC conjecture would be a publishable result.

Classification by coclass

We can classify the groups of order 2^n and nilpotency class $n-1$.
For $n > 3$ there are three isomorphism classes.
So let us extraplate from this firm base, and conjecture:
For every prime p, and for every integer r, there is a simple formula for the number of isomorphism classes of groups of order p^n and class $n-r$, that can be read off from an examination of all groups of this kind subject to $n < k$, where k is a function of p and r.

If P is any group of order p^n and class c we define the *coclass* of P to be $n-c$.

If $p = 2$ the above can be achieved.
We can (in a sense) classify all 2-groups by coclass up to isomorphism.

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Step 5. Reflect that a solution that depends on homological algebra and representation theory won't work.

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Do not attempt to solve the homological problem. Given a proof that it is independent of n for $n \geq 4$ it suffices to classify the groups of order 2^4 and class 3.
Done

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Step 2 (new) define a p-group P of coclass r to be a *main line* group if P is a (sufficiently large) quotient of a pro-p group of coclass r.
There is a function $f(r)$ such that, if P is a (sufficiently large) 2-group of coclass r then P has a normal abelian subgroup A of order dividing $2^{f(r)}$, such that P/A is a main line group.

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Final step. The homological problem to be solved is periodic, with period 2^{r-1}.