Codegrees and nilpotence class of p-groups

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Joint work with Ni Du

This talk is dedicated in memory of David Chillag

Introduction

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We go a different direction.

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Surprisingly, Qian did the solvable case and Isaacs did the nonsolvable case.

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If N is a normal subgroup of G, then $\mathrm{cod}(G/N) \subseteq \mathrm{cod}(G)$.

Goal: Show that we can bound the nilpotence class of a p-group G in terms of $\mathrm{cod}(G)$.

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Corollary 3.

If G is a nontrivial p-group for some prime p, then $p \in \mathrm{cod}(G)$.

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Let G be a group and let p be a prime. Then $\mathrm{cod}(G) = \{1, p\}$ if and only if G is a nontrivial elementary abelian p-group.

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Since $\mathrm{cod}(G/G') \subseteq \mathrm{cod}(G)$, this implies that $p^2 \notin \mathrm{cod}(G/G')$. By Lemma 2, it follows that $\mathrm{cod}(G/G') = \{1, p\}$, and applying Lemma 4, we conclude that G/G' is elementary abelian. □

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If G has a faithful irreducible character, then $|G|$ can be bounded in terms of $\mathrm{cod}(G)$.

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Let G be a group and suppose that $\chi \in \mathrm{Irr}(G)$ is faithful. Then $|G| = \chi(1)\mathrm{cod}(\chi) < \mathrm{cod}(\chi)^2$.

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If G is a p-group and $p < p^a = \max(\mathrm{cod}(G))$, then G has nilpotence class at most $2a - 2$.

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Thus, we may assume $\chi \in \mathrm{Irr}(G)$ is faithful.
By Lemma 6, we have $|G| < p^{2a}$.
This implies that $|G| \leq p^{2a-1}$, and G has nilpotence class at most $2a - 2$. □

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Notice that if G is an extraspecial p-group of order p^3, then $p^2 = \max(\mathrm{cod}(G))$ and G has nilpotence class $2 = 2 \cdot 2 - 2$, so the bound is sharp when $a = 2$.

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When $p = 3$, we use computer algebra system Magma to find examples of group having $3^3 = \max(\mathrm{cod}(G))$ and G having nilpotence class 4.

In particular, this holds for $\mathrm{SmallGroup}(3^5, i)$ for $i = 28, 29, 30$.

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Recall that a p-group G is said to have maximal class if $|G| = p^n$ where $n > 2$ is an integer and G has nilpotence class $n - 1$.

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Recall that a p-group G is said to have maximal class if $|G| = p^n$ where $n > 2$ is an integer and G has nilpotence class $n - 1$.

Lemma 8.

Suppose G is a p-group. Assume $|G|$ is minimal so that $p < p^a = \max(\mathrm{cod}(G))$ and G has nilpotence class $2a - 2$. Then $|G| = p^{2a-1}$, G has maximal class, and $p^{a-1} \in \mathrm{cd}(G)$. If $a > 3$, then G' is nonabelian.

We use Lemma 8 to see that the bound in Lemma 7 can be improved when $p = 2$ and $a > 2$ and when $p = 3$ and $a > 3$.

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Corollary 9.

Let G be a p-group, and assume $p^a = \max(\mathrm{cod}(G))$. Assume either $p = 2$ and $a > 2$ or $p = 3$ and $a > 3$. Then G has nilpotence class at most $2a - 3$.

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Then we can find G, a p-group with $p^a = \max(\mathrm{cod}(G))$ and having nilpotence class $2a - 2$ where $|G|$ is minimal.

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We may use Lemma 8 to see that G has maximal class, $p^{a-1} \in \mathrm{cd}(G)$, and is not metabelian if $a > 3$.
If $p = 2$, then we know that $\mathrm{cd}(G) = \{1, 2\}$ when G has maximal class, so we have a contradiction when $a > 2$.

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If $p = 2$, then we know that $\mathrm{cd}(G) = \{1, 2\}$ when G has maximal class, so we have a contradiction when $a > 2$.
When $p = 3$ and $a > 3$, this is a contradiction since it is known that 3-groups with maximal class are metabelian. $\square$

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Also, it seems likely that if $p = 5$ and $a > 4$, then G has nilpotence class at most $2a - 3$.

Question: For what primes p and values a, is the bound $2a - 2$ sharp for the nilpotence class of a p-group G with $p^a = \max(\mathrm{cod}(G))$?

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An example of a group G with $\max(\mathrm{cod}(G)) = 2^4$ and nilpotence class 4 is $\mathrm{SmallGroup}(2^7, 138)$.

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An example of a group G with $\max(\mathrm{cod}(G)) = 2^4$ and nilpotence class 4 is $\mathrm{SmallGroup}(2^7, 138)$.

For $p = 3$, the group $\mathrm{SmallGroup}(3^7, 226)$ is an example of a group G having $\max(\mathrm{cod}(G)) = 3^4$ and G has nilpotence class 4.

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If G is a p-group and $|\mathrm{cod}(G)| = 3$, then G has nilpotence class at most 2.

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1. Can we bound the nilpotence class of a p-group G in terms of $|\mathrm{cod}(G)|$ when $|\mathrm{cod}(G)| \geq 4$?

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2. Can we bound the derived length of G in terms of $\mathrm{cod}(G)$?

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This implies that G/G' is elementary abelian.

We may assume that G has a faithful irreducible character χ.

We show $\mathrm{cod}(\chi) = p^a$ and $|G| = \chi(1)p^a \leq p^{2a-1}$.

Extra slide: Proof

Sketch of proof of Theorem 10:

If $\mathrm{cod}(G) = \{1, p, p^2\}$, then G has nilpotence class at most $2 \cdot 2 - 2 = 4 - 2 = 2$.

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We may assume that G has a faithful irreducible character χ.

We show $\mathrm{cod}(\chi) = p^a$ and $|G| = \chi(1)p^a \leq p^{2a-1}$.

Let Z be the center of G.

Extra slide: Proof

We may assume that G/Z is nonabelian.

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By induction, we may assume that G/Z has nilpotence class 2.

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Let $K = \ker(\psi)$ and $Y = Z(\psi)$.

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This implies that $\mathrm{cod}(\psi) = p^a$.

Let $K = \ker(\psi)$ and $Y = Z(\psi)$.

$p^a = |G : K|/\psi(1)$, so $|G : K| = p^a\psi(1)$.

Extra slide: Proof

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We conclude that $\chi(1) = \psi(1)|K|$.

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Since G/K has nilpotence class 2, we have $\psi(1)^2 = |G:Y|$.

$p^a\psi(1) = |G:K| = |G:Y||Y:K| = \psi(1)^2|Y:K|$.

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$\chi(1)\psi(1) = |G:Y||K| = \psi(1)^2|K|$.

We conclude that $\chi(1) = \psi(1)|K|$.

Since $\chi(1) < p^a$, we have $\psi(1)|K| < p^a = \psi(1)|Y:K|$.

Extra slide: Proof

Since G/K has nilpotence class 2, we have $\psi(1)^2 = |G : Y|$.

$p^a \psi(1) = |G : K| = |G : Y||Y : K| = \psi(1)^2 |Y : K|$.

This implies that $p^a = \psi(1)|Y : K|$.

$|G| = \chi(1)p^a = \chi(1)\psi(1)|Y : K|$.

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We conclude that $\chi(1) = \psi(1)|K|$.

Since $\chi(1) < p^a$, we have $\psi(1)|K| < p^a = \psi(1)|Y : K|$.

We determine that $|K| < |Y : K|$.

Extra slide: Proof

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G/G' is elementary abelian implies $G'/[G', G]$ is elementary abelian.

This implies that $G'K/K \cong G'/K \cap G'$ is elementary abelian.

Also, $G'K/K$ is cyclic.

Extra slide: Proof

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There is an element y so that $Y = \langle y, Z \rangle$.

This implies that $G' = \langle y^p, Z \rangle$.

If $g \in G$, then $[y, g] \in [Y, G] \leq K = Z$.

Extra slide: Proof

Since Z has order p, it follows that $[y^p, g] = [y, g]^p = 1$.

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We conclude that $[G', G] = 1$, and so, $G' \leq Z$ which is a contradiction.