Codegrees and nilpotence class of $p$-groups

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Joint work with Ni Du
This talk is dedicated in memory of David Chillag
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We know if $G$ is solvable, then $dl(G)$ can be bounded in terms of $|cd(G)|$. 
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In particular, for every prime $p$ and positive integer $n$, there exists a $p$-group $G$ with $cd(G) = \{1, p\}$ and nilpotence class $n$. 
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We go a different direction.
We define the codegree of $\chi$ to be $\text{cod}(\chi) = |G : \ker(\chi)|/\chi(1)$. 
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Qian and Isaacs have shown that if $g \in G$, then there exists $\chi \in \text{Irr}(G)$ such that every prime that divides $o(g)$ divides $\text{cod}(\chi)$. 
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Surprisingly, Qian did the solvable case and Isaacs did the nonsolvable case.
We define $\text{cod}(G) = \{\text{cod}(\chi) \mid \chi \in \text{Irr}(G)\}$. 

One advantage of our definition: If $N$ is a normal subgroup of $G$, then $\text{cod}(G/N) \subseteq \text{cod}(G)$.
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One advantage of our definition:

If $N$ is a normal subgroup of $G$, then $\text{cod}(G/N) \subseteq \text{cod}(G)$.

Goal: Show that we can bound the nilpotence class of a $p$-group $G$ in terms of $\text{cod}(G)$.
Basics

We collect several basic results regarding codegrees.
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**Lemma 1.**

Let $G$ be a group, and let $\chi \in \text{Irr}(G)$. If $\chi \neq 1_G$, then $\chi(1) < \text{cod}(\chi)$. 
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**Lemma 1.**

Let $G$ be a group, and let $\chi \in \text{Irr}(G)$. If $\chi \neq 1_G$, then $\chi(1) < \text{cod} (\chi)$.

**Proof.**

Since $\chi \neq 1_G$, we see that $G/\ker(\chi) > 1$. 
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It follows that

$\text{cod}(\chi) = |G: \ker(\chi)|/\chi(1) > \chi(1)^2/\chi(1) = \chi(1)$. 

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Codegrees of $p$-groups
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**Lemma 2.**

Let $G$ be a group, let $e$ be the exponent of $G/G'$, and let $d$ be a divisor of $e$. Then $d \in \text{cod}(G)$. 
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**Lemma 2.**

Let $G$ be a group, let $e$ be the exponent of $G/G'$, and let $d$ be a divisor of $e$. Then $d \in \text{cod}(G)$.

**Corollary 3.**

If $G$ is a nontrivial $p$-group for some prime $p$, then $p \in \text{cod}(G)$. 
Lemma 4.

Let $G$ be a group and let $p$ be a prime. Then $\text{cod}(G) = \{1, p\}$ if and only if $G$ is a nontrivial elementary abelian $p$-group.
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**Proof.**

Since $\text{cod}(G/G') \subseteq \text{cod}(G)$, this implies that $p^2 \notin \text{cod}(G/G')$. By Lemma 2, it follows that $\text{cod}(G/G') = \{1, p\}$, and applying Lemma 4, we conclude that $G/G'$ is elementary abelian.
Bounding nilpotence class

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Recall that \( \chi \in \text{Irr}(G) \) is faithful if \( \ker(\chi) = 1 \).

If \( G \) has a faithful irreducible character, then \( |G| \) can be bounded in terms of \( \text{cod}(G) \).
Lemma 6.

Let $G$ be a group and suppose that $\chi \in \text{Irr}(G)$ is faithful. Then $|G| = \chi(1)\text{cod}(\chi) < \text{cod}(\chi)^2$. 
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Recall that if $|G| = p^a$, then $G$ has nilpotence class at most $a - 1$.

**Lemma 7.**

*If $G$ is a $p$-group and $p < p^a = \max(\text{cod}(G))$, then $G$ has nilpotence class at most $2a - 2$.***
Proof.

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If $\chi \in \text{Irr}(G)$ is not faithful, then $\max(\text{cod}(G/\ker(\chi))) \leq p^a$.

By the inductive hypothesis, $G/\ker(\chi)$ has nilpotence class at most $2^a - 2$.

If none of the irreducible characters of $G$ are faithful, then $G$ has nilpotence class at most $2^a - 2$.

Thus, we may assume $\chi \in \text{Irr}(G)$ is faithful.

By Lemma 6, we have $|G| < p^{2^a}$.

This implies that $|G| \leq p^{2^a - 1}$, and $G$ has nilpotence class at most $2^a - 2$. 
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\[\square\]
Is the bound sharp?

It would be interesting to see when the bound in Lemma 7 is sharp.
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Notice that if $G$ is an extraspecial $p$-group of order $p^3$, then $p^2 = \max(\text{cod}(G))$ and $G$ has nilpotence class $2 = 2 \cdot 2 - 2$, so the bound is sharp when $a = 2$. 
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When $p = 3$, we use computer algebra system Magma to find examples of group having $3^3 = \max(\text{cod}(G))$ and $G$ having nilpotence class 4.
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When $p = 3$, we use computer algebra system Magma to find examples of group having $3^3 = \max(\text{cod}(G))$ and $G$ having nilpotence class 4.

In particular, this holds for $\text{SmallGroup}(3^5, i)$ for $i = 28, 29, 30$. 
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$G = \text{SmallGroup}(5^7, i)$ for $1306 \leq i \leq 1310$ and $1358 \leq i \leq 1380$. 
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Recall that a $p$-group $G$ is said to have maximal class if $|G| = p^n$ where $n > 2$ is an integer and $G$ has nilpotence class $n - 1$. 

Lemma 8. Suppose $G$ is a $p$-group. Assume $|G|$ is minimal so that $p < p^a = \max(\text{cod}(G))$ and $G$ has nilpotence class $2a - 2$. Then $|G| = p^{2a - 1}$, $G$ has maximal class, and $p^{a - 1} \in \text{cd}(G)$. If $a > 3$, then $G'$ is nonabelian.
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We use Lemma 8 to see that the bound in Lemma 7 can be improved when $p = 2$ and $a > 2$ and when $p = 3$ and $a > 3$. 
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1. If $p = 2$, then $cd(G) = \{1, 2\}$.
2. If $p = 3$, then $G$ must be metabelian.

Corollary 9. Let $G$ be a $p$-group, and assume $p = \max(cod(G))$. Assume either $p = 2$ and $a > 2$ or $p = 3$ and $a > 3$. Then $G$ has nilpotence class at most $2a - 3$. 
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Let $G$ be a $p$-group, and assume $p^a = \max(\text{cod}(G))$. Assume either $p = 2$ and $a > 2$ or $p = 3$ and $a > 3$. Then $G$ has nilpotence class at most $2a - 3$. 
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Proof.

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If $p = 2$, then we know that $\text{cd}(G) = \{1, 2\}$ when $G$ has maximal class, so we have a contradiction when $a > 2$.

When $p = 3$ and $a > 3$, this is a contradiction since it is known that 3-groups with maximal class are metabelian.
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Also, it seems likely that if $p = 5$ and $a > 4$, then $G$ has nilpotence class at most $2a - 3$. 
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Also, it seems likely that if $p = 5$ and $a > 4$, then $G$ has nilpotence class at most $2a - 3$.

Question: For what primes $p$ and values $a$, is the bound $2a - 2$ sharp for the nilpotence class of a $p$-group $G$ with $p^a = \max(\text{cod}(G))$?
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Using Magma, we find examples where $\max(\text{cod}(G)) = 2^3$ and $G$ has nilpotence class 3.
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One such example is SmallGroup($2^5$, 6).
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Using Magma, we find examples where \( \max(\text{cod}(G)) = 2^3 \) and \( G \) has nilpotence class 3.

One such example is \( \text{SmallGroup}(2^5, 6) \).

An example of a group \( G \) with \( \max(\text{cod}(G)) = 2^4 \) and nilpotence class 4 is \( \text{SmallGroup}(2^7, 138) \).
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One such example is $\text{SmallGroup}(2^5, 6)$.

An example of a group $G$ with $\max(\text{cod}(G)) = 2^4$ and nilpotence class 4 is $\text{SmallGroup}(2^7, 138)$.

For $p = 3$, the group $\text{SmallGroup}(3^7, 226)$ is an example of a group $G$ having $\max(\text{cod}(G)) = 3^4$ and $G$ has nilpotence class 4.
Question:
Introduction Basics A bound on the nilpotence class Is the bound sharp? Another bound

Question:
Is the bound $2a - 3$ sharp for $a \geq 5$ when $p = 2$ or 3 and $p^a = \max(\text{cod}(G))$?
Another bound

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**Theorem 10.**

*If $G$ is a $p$-group and $|\text{cod}(G)| = 3$, then $G$ has nilpotence class at most 2.*
Another bound

We conclude with a small piece of evidence that perhaps the nilpotence of $G$ is also bounded in terms of $|\text{cod}(G)|$.

**Theorem 10.**

*If $G$ is a $p$-group and $|\text{cod}(G)| = 3$, then $G$ has nilpotence class at most 2.*

Questions:

1. Can we bound the nilpotence class of a $p$-group $G$ in terms of $|\text{cod}(G)|$ when $|\text{cod}(G)| \geq 4$?
Another bound

We conclude with a small piece of evidence that perhaps the nilpotence of $G$ is also bounded in terms of $|\text{cod}(G)|$.

**Theorem 10.**

*If $G$ is a $p$-group and $|\text{cod}(G)| = 3$, then $G$ has nilpotence class at most 2.*

Questions:

1. Can we bound the nilpotence class of a $p$-group $G$ in terms of $|\text{cod}(G)|$ when $|\text{cod}(G)| \geq 4$?

2. Can we bound the derived length of $G$ in terms of $\text{cod}(G)$?
Sketch of proof of Theorem 10:

If \( \text{cod}(G) = \{1, p, p^2\} \), then \( G \) has nilpotence class at most \( 2 \cdot 2 - 2 = 2 \).

We know that \( p \in \text{cod}(G) \).

We may assume that \( \text{cod}(G) = \{1, p, p^a\} \) where \( a > 2 \).

This implies that \( G/G' \) is elementary abelian.

We may assume that \( G \) has a faithful irreducible character \( \chi \).

We show \( \text{cod}(\chi) = p^a \) and \( |G| = \chi(1) p^a \leq p^{2a - 1} \).

Let \( Z \) be the center of \( G \).
Extra slide: Proof

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Let $K = \ker(\psi)$ and $Y = Z(\psi)$.

\[ p^a = \frac{|G : K|}{\psi(1)}, \text{ so } |G : K| = p^a\psi(1). \]
Since $G/K$ has nilpotence class 2, we have $\psi(1)^2 = |G : Y|$.
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$p^a \psi(1) = |G : K| = |G : Y||Y : K| = \psi(1)^2 |Y : K|$. 

This implies that $p^a = \psi(1) |Y : K|$. 

We conclude that $\chi(1) = \psi(1) |K|$. 

Since $\chi(1) < p^a$, we have $\psi(1) |K| < p^a = \psi(1)|Y : K|$. 

We determine that $|K| < |Y : K|$. 

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Codegrees of $p$-groups
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Since $\chi(1) < p^a$, we have $\psi(1)|K| < p^a = \psi(1)|Y : K|.$

We determine that $|K| < |Y : K|.$
ψ is a faithful character of $G/K$, so $Y/K$ must be cyclic.
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Since $G/G'$ is elementary abelian, $Y/G'K$ is elementary abelian.

This implies that $Y/G'K$ is both elementary abelian and cyclic, so $|Y : G'K|$ divides $p$. 
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$G/G'$ is elementary abelian implies $G' / [G', G]$ is elementary abelian.
Extra slide: Proof

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This implies that \( Y/G'K \) is both elementary abelian and cyclic, so \( |Y : G'K| \) divides \( p \).

\( G/G' \) is elementary abelian implies \( G'/[G', G] \) is elementary abelian.

This implies that \( G'K/K \cong G'/K \cap G' \) is elementary abelian.
\( \psi \) is a faithful character of \( G/K \), so \( Y/K \) must be cyclic.

Since \( G/G' \) is elementary abelian, \( Y/G'K \) is elementary abelian.

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\( G/G' \) is elementary abelian implies \( G'/[G', G] \) is elementary abelian.

This implies that \( G'K/K \cong G'/K \cap G' \) is elementary abelian.

Also, \( G'K/K \) is cyclic.
This implies that $|G'K : K|$ divides $p$. 
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We obtain $p \leq |K| < |Y : K| \leq p^2$. 
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We obtain $p \leq |K| < |Y : K| \leq p^2$.

There is an element $y$ so that $Y = \langle y, Z \rangle$. 

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Codegrees of $p$-groups
This implies that $|G'K : K| \text{ divides } p$.

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This implies that $|G'K : K|$ divides $p$.

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We obtain $p \leq |K| < |Y : K| \leq p^2$.

There is an element $y$ so that $Y = \langle y, Z \rangle$.

This implies that $G' = \langle y^p, Z \rangle$.

If $g \in G$, then $[y, g] \in [Y, G] \leq K = Z$. 
Since $Z$ has order $p$, it follows that $[y^p, g] = [y, g]^p = 1$. 
Since $Z$ has order $p$, it follows that $[y^p, g] = [y, g]^p = 1$.

We conclude that $[G', G] = 1$, and so, $G' \leq Z$ which is a contradiction.