Deformation Theory and Finite Simple Quotients of Triangle Groups

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2 papers (see Arxiv) to appear in JEMS & GGD.

I. Hyperbolic triangle gps

$2 \le a \le b \le c \in \mathbb{N}$ w. $\mu = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

$T = T_{a,b,c} = \langle x, y, z \mid x^a = y^b = z^c = xyz = 1 \rangle$

$\mathbb{H}^2 \circlearrowleft T$, $S = {}_{\Gamma}\backslash \mathbb{H}^2$ - Riemann surfac

Siegel min covolume $T_{2,3,7}$.

Hurwitz $\quad Aut(S) \le 84(g-1)$, g = genus

Equality $\Leftrightarrow \pi_1(S) \triangleleft T_{2,3,7}$ in which case $Aut(S) = T_{2,3,7} / \pi_1(S)$

Def Hurwitz gp $\equiv$ quotient of $T_{2,3,7}$

Question: What are the (finite/simple) Hurwitz gps?

Many pos. and neg. results by explicit or probablistic method. (Macbeath, ..., Tamburini)

Probablistic: (a) Estimate $\# \mathrm{Hom}(T, G)$ usually using character theory.

(b) Estimate how many of them are epi, but the knowledge of max subgps of FSG.

Q. (Guralnick) Is $E_8(q)$ quotient of $T_{2,3,7}$, for some q ??

Main goal today:
Presenting a 3rd method for these questions based on deformation theory

Starting point: Claude Maion's thesis; he tried to draw some general principles to explain and extend the known results on FSQ quotients of general $T_{a,b,c}$.

His invariants ......

Deformation Theory

~~I~~ Representation Varieties

F alg. closed field, e.g. $F = \mathbb{C}$

G a (simple) alg. gp over F.

$\Gamma = \langle x_1, \dots, x_d \mid R_1, \dots, R_r \dots \rangle$ a f.g. gp

$R(\Gamma, G) = \mathrm{Hom}(\Gamma, G(F))$ is an (affine) alg variety. $\rho \longleftrightarrow (\rho(x_1), \dots, \rho(x_d)) \in G^d$

$X(\Gamma, G) = \underline{\text{character variety}} =$
$= R(\Gamma, G) /\!\!/ G$

$[\rho] \in X(\Gamma, G)$, $\underline{\text{Tangent space}}$ at ρ is inside $\underline{H^1(\Gamma, \mathrm{Ad}\circ\rho|_{\mathfrak{g}})}$, $\mathfrak{g} = \underline{\text{Lie alg.}}$

- For ρ "nice" $\rightsquigarrow$ equality.

<u>Def</u> Γ is <u>locally rigid</u> at $\rho \in R(\Gamma, G)$ if $[\rho]$ is an isolated pt (i.e. no non trivial deformations).

This + computations of $H^1(T_{a,b,c}, \mathfrak{g})$ by A. Weil lead to prove a conj. of <u>Marion</u> which is on the neg. side: ....

But also <u>positive results</u>: (with a method which applies to <u>any</u> Γ)

Positive results

Def Γ f.g. group, X a Dynkin diagram, G - Chevalley gp of type X: $X(q) = G(q)/Z$

Say: Γ is **saturated with quotients of type X** if $\exists\, p_0, e \in \mathbb{N}$ s.t. $\forall$ prime $p > p_0$, $X(p^{e\ell})$ is a quotient of Γ, $\forall \ell$ and for a pos. dense set of prime $X(p^{\ell})$ is so $\forall \ell$.

<u>General Thm.</u> (Larsen, L., Mation)
Γ f.g., X - Dynkin diagram

T.F.A.E.

(1) $\exists$ a rep $\rho : \Gamma \to X(\mathbb{C})$ with Zariski dense image s.t. ρ is <u>not</u> loc. rigid in $\mathrm{Hom}(\Gamma, X(\mathbb{C}))$.

(2) Γ is saturated with finite quotients of type X.

Explain via Strong Aproximation

The game now moves to Char. 0.

We developed few methods to produce such <u>Zariski dense non loc. rigid rep's</u> (some robust/some special)

We deduce:

<u>Theorem</u> (Larsen, L., Marion)

$T_{a,b,c}$ is saturated by quotients of type X unless appear in Table 1 or 2

Possible Exceptions

X	(a,b,c)	r
A_r	$(2,3,7)$	$5 \le r \le 19$
	$(2,3,8)$	$5 \le r \le 13$
	$(2,3,c)$ $c \ge 9$	$5 \le r \le 7$
	$(2,4,5)$	$3 \le r \le 13$
	$(2,4,6)$	$3 \le r \le 9$
	$(2,4,c)$ $c \ge 7$	$3 \le r \le 5$
	$(2,5,5)$	$r = 6$
	$(2,b,c)$ $b \ge 5$, $c \ge 5$	$r = 3$
	$(3,3,c)$ $c \ge 4$	$r \in \{3,4,6\}$
B_3	$(2,3,c)$ $c \ge 7$	
	$(3,3,c)$ $c \ge 4$, $c \neq 15c_1$	
	$(2,4,5)$	
	$(2,5,5)$	
D_r	$(2,3,7)$	$r \in \{4,5,9\}$
	$(2,3,8)$	$\times 3$ $r \in \{4,5\}$
	$(2,3,9)$	$r \in \{4,5\}$
	$(2,3,10)$	
	$(2,3,c)$ $c \ge 11$, $c \neq 15c_1$	$r = 4$
	$(2,3,c)$ $c \ge 12$, $c \neq 11c_1$	$r = 5$
	$(2,4,5)$	$r \in \{4,5\}$
	$(3,3,4)$	$r = 4$.
	$(3,3,c)$ $c \ge 5$ and $c \notin \{7c_1, 9c_1, 10c_1, 12c_1, 15c_1\}$	
E_6	$(2,3,7)$, $(2,3,8)$, $(2,4,5)$	
	$(2,4,6)$, $(2,4,7)$, $(2,4,8)$	

Rigid Exceptions

X	(a, b, c)
A_1	any
A_2	$a = 2$
A_3	$a = 2, b = 3$
A_4	$a = 2, b = 3$
C_2	$b = 3$
G_2	$a = 2, c = 5$

Corollaries

(1) If $\mu = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2}$ then $\forall\, X \neq A_1$, $T_{a,b,c}$ is saturated by X.

(2) Let $Y = \{A_r, 1 \leq r \leq 19\} \cup \{B_3\}$
$\cup \{C_2\} \cup \{G_2\} \cup \{E_6\} \cup$
$\cup \{D_r : r = 4, 5, 9\}$

Then for every (a,b,c) and every $X \in Y$, $T_{a,b,c}$ is saturated with X.

<u>Cor</u> <u>Every</u> $T_{a,b,c}$ (including $T_{2,3,7}$) is saturated with quotients of type E_8