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CONJUGACY CLASSES IN FINITE p-GROUPS

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If G is a finite p-group, the sizes of its conjugacy classes are powers of p. This is essentially the only restriction on these sizes, as is seen from

Theorem 1 (J.Cossey - T.O.Hawkes [CH]). *Given any finite set $\mathcal{S}$ of powers of p, including 1, there exists a p-group whose conjugacy class sizes are exactly the members of $\mathcal{S}$.*

The groups constructed by Cossey and Hawkes are of nilpotency class 2.

Problem 1. *Find other constructions, in particular ones that produce groups of higher class.*

Of course, in that problem we have to take into account that the class sizes impose restrictions on the group structure. E.g. if the sizes are $\{1, p\}$, then the nilpotency class has to be 2. More precisely: the class sizes of a p-group G are $\{1, p\}$ iff $|G'| = p$ (Knoche; see also Theorem 3 below). But we can ask, e.g., if, given any set $\mathcal{S} \neq \{1, p\}$ of p-powers, does there exist a group of class 3 whose class sizes are the members of $\mathcal{S}$.

Given an element $x \in G$ whose class size is p^b, we say that $b = b_G(x) = b(x)$ is the *breadth* of x. The *breadth* $b(G)$ of G is the maximal breadth of its elements. There is much interest in the relation of this invariant to other invariants of G which measure its departure from commutativity. The following is obvious.

Proposition 2. *If $|G'| = p^k$ and $|G : Z(G)| = p^z$, then $b(G) \leq k$ and $b(G) \leq z-1$.*

Equality is possible in both inequalities, and one of them has a converse.

Theorem 3 (M.R.Vaughan-Lee [VL]). *If $b(G) = b$ and $|G'| = p^k$, then $k \leq b(b+1)/2$.*

Again equality is possible. There is no bound for $|G : Z(G)|$ in terms of $b(G)$, consider extraspecial groups. But a bound on $|G'|$ imposes a bound on $|G : Z_2(G)|$. For explicit estimates see, e.g., [PS].

It follows from Theorem 3 that the nilpotency class $cl(G)$ is bounded in terms of $b(G)$, but that theorem does not yield the best bound. For a long time many people believed the following

Class - breadth conjecture. *A group of breadth b and class c satisfies $c \leq b+1$.*

This holds, e.g., if either the breadth is at most $p+1$, or if the class is at most $p+3$, or if G is metabelian, and in various other cases. In any case, a linear bound holds.

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Theorem 4 (C.R.Leedham-Green–P.M.Neumann–J.Wiegold [LGNW]).

$$cl(G) \leq \frac{p}{p-1} b(G) + 1.$$

This is proved by a nice counting argument, introducing the important notion of *2-step centralizers*. Let $\gamma_i(G)$ be the ith member of the upper central series of G. The subgroup $C_i(G) = C_G(\gamma_i(G)/\gamma_{i+1}(G))$ is the ith 2-step centralizer of G. There are $c-1$ such centralizers (here $c = cl(G)$), and they are proper subgroups of G. Suppose that $x \notin C_{c-1}(G) = C_G(\gamma_{c-1}(G))$. Since $\gamma_{c-1}(G)/\gamma_c(G) \leq Z(G/\gamma_c(G)) \leq C_{G/\gamma_c(G)}(x\gamma_c(G))$, we obtain $b(x\gamma_c(G)) < b(x)$. Now suppose that $x \notin \cup C_i(G)$. Then looking successively in the factor groups $G/\gamma_i(G)$, we see that $b(x) \geq cl(G)-1$. This implies the class breadth conjecture, provided we can find an appropriate x. If $cl(G) \leq p+1$, then the number of 2-step centralizers is at most p, and since a p-group cannot be the union of p proper subgroups, there is an element x as required. An easy application of the three subgroups lemma shows that $C_1 \leq C_i$, for all i, and therefore C_1 can be omitted from the above considerations. This shows that the class breadth conjecture holds if $cl(G) \leq p+2$, and some elaboration of the argument yields the other cases mentioned above. In the general case we cannot ensure that x exists, but, using the fact that we are dealing with proper subgroups, the authors of [LGNW] estimate the average number of 2-step centralizers containing each element, and from this they estimate the average of $b(x)$ and deduce Theorem 4.

For $p = 2$ the inequality can be improved

Proposition 5 (M.Cartwright [C]). $cl(G) \leq \frac{5}{3} b(G) + 1$.

The class-breadth conjecture was eventually disproved by V.Felsch [F], using computer calculations to construct a counter example of order 2^{34}, class 29, and breadth 27. Moreover, W.Felsch et al constructed 2-groups in which the difference $c-b$ can be arbitrarily large [FNP]. In these examples c is about $b+\sqrt{b}$. No counter examples for odd primes are known.

Problem 2. *Construct counter examples for odd primes (alternatively, prove that they do not exist).*

The nilpotency class can be bounded under weaker assumptions than in Theorem 4.

Theorem 6 (C.R.vaughan-Lee - J.Wiegold [VLW]). *If G is generated by elements of breadth at most b, then $cl(G) \leq b^2 + 1$.*

The author has improved the bound slightly, to $cl(G) \leq b^2 - b + 1$ (provided that $b > 1$) [M2].

Problem 3. *Can the bounds in Theorems 4 to 6 be significantly improved?*

Let us go back in history. In the 1950's N.Ito initiated a series of papers discussing finite groups with a small number of conjugacy class sizes. In particular, if all non-central classes have the same size, then Ito showed that $G \cong P \times A$, where P is a p-group and A is abelian [It]. That focuses the problem on p-groups, for which Ito proved the existence of a normal abelian subgroup N such that G/N has exponent p. This was improved by Isaacs [Is1], who showed that actually $exp(G/Z(G)) = p$

(this was reproved later, in ignorance of Isaacs and of each other, by both the author [M1] and L.Verardi [V]). Groups of exponent 2 are abelian, and the ones of exponent 3 have nilpotency class at most 3, and thus Isaacs' result implies that for $p = 2$ the class of G is 2, and for $p = 3$ the class is at most 4, but for larger primes we cannot say much, because groups of prime exponent are quite difficult to understand. Then Ishikawa made a break-through by proving

Theorem 7 (K.Ishikawa [I]). *If all non-central classes of G have the same size, then $cl(G) \leq 3$.*

This is best possible for odd primes, it is easy to construct, e.g., for each odd prime p a group of order p^5, class 3, and class sizes 1 and p^2. The proof of Theorem 7 actually shows that it suffices to assume that G is generated by its non-central classes of minimal size. We call these classes, and their elements *minimal* classes and elements. The author generalized Theorem 7 to

Theorem 8 (A.Mann [M4]). *Let G be a p-group, and let $M(G)$ be the subgroup that is generated by all the minimal elements. Then $cl(M(G)) \leq 3$.*

The proof is independent of Ishikawa's, and provides a shorter and simpler proof of his result. Moreover, while Theorem 7 deals with a severely restricted class of groups, Theorem 8 states a property of all p-groups. Following the proof of Theorem 8, I made the following conjecture (it certainly occurred also to other authors). Let the conjugacy classes of G have sizes $n_1 = 1 < n_2 = p^s < ... < n_t = p^{b(G)}$.

Conjecture A. *Let G be a finite p-group, and let the numbers t, n_i be as above. Then there exists a function $f(r)$ such that the subgroup H_r of G generated by the classes of sizes $n_1, ..., n_r$ has derived length $dl(H_r)$ at most $f(r)$.*

The conjecture implies, in particular, that $dl(G)$ is bounded by $f(t)$.

Note that if $t \geq 3$, we cannot bound $cl(G)$. Consider a non-abelian group containing an abelian maximal group. Then $t = 3$, but there are such groups of arbitrarily high class. One motivation for the conjecture is the fact that the "dual" claim, obtained by replacing class sizes by irreducible character degrees, holds: let N_r be the intersection of the kernels of the irreducible characters of G of the r smallest degrees. Then $dl(G/N_r) \leq r$. This is even true for all soluble groups, with bound $2r$, and conjecturally with much better bounds.

The conjecture was recently proved by Bettina Wilkens.

Theorem 9 [W]. *With the notation of Conjecture A, if $r \geq 2$ (i.e. G is not abelian), then $dl(H_r) \leq 2r - 2$.*

As always, we now ask if we can improve the upper bound. This is known in some very special cases.

Proposition 10 ([M3], [M4], [M5]). *Let $p = 2$. Then $cl(H_3 \cap G^2) \leq 3$ and $dl(H_3) \leq 3$; moreover, if $t = 3$ then G is metabelian, and if $t = 4$, then $dl(G) \leq 3$, and generally $dl(G) \leq 2t - 3$.*

There are examples of groups with $t = 4$ and derived length 3, but these constructions are for $p \geq 5$ [IM].

One of the difficulties in proving the conjecture and related results is that induction is often not available, because the number t can increase when we move from G to a subgroup or a factor group. The key to proving Theorem 8 was concentrating on the breadth of one element, rather than of the full group. Take an element $x \in G$. Since $G_G(x) \leq C_G(x^p)$, we have $b(x^p) \leq b(x)$, and it is to be expected that usually the inequality is strict. Of course, this need not always be the case. If memory serves, I have heard from K.Harada, discussing the classification of the finite simple groups, the dictum: concentrate all the bad things in one place. Thus we make the

Definition. The *centralizer equality subgroup D(G)* of G is given by

$$D(G) = \langle x \mid x \in G,\ C_G(x^p) = C_G(x) \rangle .$$

Theorem 11 (Mann [M3]). *The centralizer equality subgroup is abelian.*

This is rather surprising, because we do not expect distinct elements with the defining property of $D(G)$ to be related to each other. Nevertheless, the proof, which was suggested by an argument in [Is1], is quite simple.

Proof. Suppose that $D(G)$ is not abelian. Then there exists an element $z \in Z_2(D(G)) - Z(D(G))$, $z^p \in Z(D(G))$. Let x be one of the defining elements of $D(G)$, and write $H = \langle x, z \rangle$. Then $cl(H) \leq 2$, implying $[x^p, z] = [x, z^p] = 1$, and thus $z \in C_G(x^p) = C_G(x)$. Therefore z commutes with all the defining elements of $D(G)$, i.e. $z \in Z(D(G))$, a contradiction.

Corollary 12. *With the notations of Conjecture A, G contains a normal abelian subgroup D such that* $exp(G/D) \leq p^{t-1}$.

Proof. For $x \in G$, among the $t+1$ elements $x, x^p, ..., x^{p^t}$ there must be two with the same class size, and therefore the same centralizer. If these elements are x^{p^i} and $x^{p^{i+1}}$, then $i \leq t-1$ and $x^{p^i} \in D(G)$.

For $p = 2$, it is possible to show that $exp(G/D) \leq 2^{t-2}$.

Another breadth diminishing device is given in the following

Proposition 13. *Let A be a normal abelian subgroup of the finite group G, let $z \in A$ and $x \in G$. If $x \notin Z(G)$, then the conjugacy class of $[x, z]$ has size smaller than that of the class of x.*

Proof. Our original proof, by induction, applied only to p-groups. The present proof is due to Isaacs [Is2] and it applies to all finite groups. First, induction shows that we may assume that $G = AC_G(x)$. Then $[A, x] \lhd G$, and $|[A, x]| > 1$ is the size of the conjugacy class of x. Since $[x, z] \in [A, x]$, all the conjugates of $[x, z]$ lie in $[A, x]$, but they do not exhaust that subgroup, hence their number is less than $|[A, x]|$.

Proof of Theorem 8. Write $N = M(G)$, and let A be maximal among the normal abelian subgroups of G that are contained in N. Then $C_N(A) = A$. If x is a minimal element, the last proposition shows that $[A, x] \leq Z(G)$. Since the minimal elements generate N, it follows that $A \leq Z_2(N)$. Then $N' \leq C_N(A) = A$, and thus $N' \leq Z_2(N)$, implying $cl(N) \leq 3$.

If $p = 2$, a separate argument shows that $cl(M(G)) \leq 2$, but for odd primes the class can be 3.

Since Proposition 13 holds for all finite groups, the conclusion of Theorem 8 holds also for many groups that are not necessarily p-groups, e.g. for supersoluble groups. For these results, see [Is2] and [M6].

sketch of a partial proof of of Proposition 10. Since $D(G)$ ia abelian, Proposition 13 shows that $[D(G), H_r(G)] \leq H_{r-1}(G)$. It follows that $dl(D(G)H_r(G)) =$█

$dl(H_r(G))$. Now $G/D(G)H_{t-1}(G)$ has exponent p. If $p=2$, the last factor group is abelian, which implies the bound $2t-3$. More argument, of a similar type, suffices for the other claims.

The proof of Theorem 9 is much more involved. It starts with a variation on Proposition 13, and proceeds with a quite elaborate argument.

There is another variation on Proposition 13.

Proposition 14. *Under the assumptions and notations of Proposition 13, assume also that G is a p-group, and let $y=[z,x,...,x]$, with k occurrences of x. If $b(x)\geq k$, then $b(y)\leq b(x)-k$.*

The proof is almost identical to that of Proposition 13, noting that the subgroups $[A,x,...,x]$, with increasing number of x's, decrease strictly till they get to the identity.

Y.Barnea and I.M.Isaacs [BI] suggested another direction for generalizations of Theorem 7. Recall that we have denoted by p^s and p^b the sizes of the smallest and largest non-central conjugacy classes. We call the difference $d=b-s$ the *class spread*, or just *spread*, of G. Barnea and Isaacs conjectured that $cl(G)$ is bounded by a function of d. This was verified by A.Jaikin-Zapirain, who gave the explicit bound $cl(G)\leq 2d^2+2d+3$ [JZ1]. This can be improved to

Theorem 15 ([M7]). *If $d\geq 1$, then $cl(G)\leq \frac{p}{p-1}d+3-\frac{1}{p-1}$.*

This is similar to Theorem 4, and is proved by combining the argument of [JZ1] with the proof of Theorem 4 in [LGNW]. The proof also handles the case $d=0$, which is Ishikawa's result. If either $c\leq p+3$ or $d\leq p-1$, then the inequality $c\leq d+3$ holds. Note that $d\leq b-1$, and substituting that value in Theorem 16 yields an inequality that is only fractionally worse than Theorem 4.

Propositions 13 and 14 suggest a different approach to the Barnea-Isaacs conjecture. They imply that if A is a normal abelian subgroup of a group of spread d, the elements of A become so called *right (d+1)-Engel elements* in $G/Z(G)$. It follows from [CT] that there exists a function $f(d)$ such that if $p>f(d)$, then a normal subgroup consisting of right d-Engel elements lies in $Z_{f(d)}(G)$. This applies then to our subgroup A, and the argument proving Theorem 8 shows that there is a bound for $cl(G)$ in terms of the gap. This approach yields a weaker result then Theorem 15, or even from the earlier result in [JZ1], where there is no restriction on p, and the bound for the nilpotent class is explicit, but it is possible that a further argument will improve the results. The two smallest values for the spread can be explicitly handled by this method. Consider the case $d=1$. For odd primes, right 2-Engel elements lie in $Z_3(G)$. It follows that $A\leq Z_4(G)$. If we take A to be a maximal normal abelian subgroup of G, then $\gamma_4(G)\leq C_G(A)=A\leq Z_4(G)$, implying $cl(G)\leq 7$. For $d=2$, a result of P.G.Crosby [Cro] states that if a normal subgroup A of a group G consists of right 3-Engel elements, and if G has no elements of orders 2,3, or 5, then $A\leq Z_8(G)$. Thus if G is a p-group, with $p\geq 7$, and of spread 2, then we obtain as above $A\leq Z_9(G)$, which implies $cl(G)\leq 17$.

Let us sketch the proof of Theorem 15. First we quote from [JZ1].

Proposition 16. *Under the assumptions and notation of Theorem 1, let $H=G/\gamma_c(G)$. There exists a proper subgroup $K<H$, such that if $v\in H-K$, then the number of conjugates of v under H' is at most p^d.*

This is equation (2.2) of [JZ1]. It is proved by a highly non-trivial adaptation of Ishikawa's argument. Jaikin proceeded as follows. An element of K can be written as a product of two elements outside K, therefore it has at most p^{2d} conjugates under H', and in particular $b(H') \leq 2d$. Theorem 3 then bounds the order of H''. A separate argument bounds the class of H/H'', and adding the bound for $|H''|$ yields the quadratic bound for $cl(G) = cl(H) + 1$. We used the method of [LGNW] to prove

Theorem 17. *Let the p-group H contain two subgroups, N and K, with N normal, such that for some natural number d, if $v \notin K$, then the number of N-conjugates of v is at most p^d. Then $N \leq Z_{[\frac{p}{p-1}d]+1}(H)$.*

Theorem 15 follows by substituting H' for N in the last theorem.[]

Let me mention still one more type of results. Using the notations preceding Conjecture A, let there be m_i classes of size n_i. The nature of these numbers is far from clear. Obviously, $m_1 = |Z(G)|$ is a power of p. The other m_i's are multiples of $p-1$, and the papers [Mc],[LMM],[M3],[JZNO] discuss the possibility of equality $m_i = p - 1$ for some i. One major result is

Theorem 18 - (Jaikin-Zapirain [JZ2]). *Given a number A, there are only finitely many p-groups for which $m_i \leq A$, for all i.*

This was extended to all finite soluble groups [JZ3], and conjecturally it holds for all finite groups, see [Ng]. The corresponding result for character degrees holds [Cr].

The sum of all the m_i's is the *class number* $k(G)$, the number of all conjugacy classes of G. P.Hall proved that

$$k(G) = 1 + e(p-1) + m(p^2-1) + a(p^2-1)(p-1)$$

for some non-negative integer $a = a(G)$ (see [M1]).

Theorem 19 - (Jaikin-Zapirain [JZ4]). *The number $a(G)$ tends to infinity with $|G|$.*

I will conclude with some variations on the above results. First, it is clear that Proposition 2 is not restricted to p-groups. In any group G (possibly infinite) the class sizes are bounded both by $|G : Z(G)|$ and by $|G'|$. It is also true that if all class sizes are bounded, then so is the order of G'. This was originally proved by B.H.Neumann, and the best bound that I know of occurs in [SS].

Next, many of our results hold also for finite-dimensional nilpotent Lie algebras, with the codimensions of centralizers replacing breadths. E.g., if we define spread in the same way, we obtain, using much the same proof

Theorem 20. *Let L be a finite-dimensional nilpotent Lie algebra over a field F, with nilpotence class c and spread d. If F is infinite, then $c \leq d+3$. If the underlying field is finite of size q, then $c \leq \frac{q}{q-1}d + 3 - \frac{1}{q-1}$.*

The other variation is for finitely generated torsion-free nilpotent groups. If G is such a group, and L a subgroup of it, we write $i(G, L) = h(G) - h(L)$, where $h(X)$ denotes the Hirsch length of X. The *breadth* of an element x is defined as $i(G, C_G(x))$, and the breadth of the group is $b(G) = \max_{x \in G} b(x)$. The *spread* $d(G)$ is the difference between $b(G)$ and the minimal breadth of non-central elements.

These notions were introduced in [MS], where various parallels of previous results were established, in particular the same bound as in [JZ1] was obtained. Here we have

Theorem 21. *Let G be a finitely generated torsion-free nilpotent group. Then $cl(G) \leq d(G) + 3$.*

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