Cyclic and pro-cyclic coverings of commutators

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Joint work with: G.A. Fernández-Alcober and P. Shumyatsky

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Observation (Baer)

A group covered by finitely many cyclic subgroups is either cyclic or finite.

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Theorem (G. Cutolo, C. Nicotera)

If G is a group in which the set of all γ_j-commutators is covered by finitely many cyclic subgroups, then $\gamma_j(G)$ is finite-by-cyclic ($\gamma_j = [x_1, x_2, \ldots, x_j]$).

Moreover $\gamma_j(G)$ can be infinite and not cyclic.

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Moreover $\gamma_j(G)$ can be infinite and not cyclic.
It is still unknown whether a similar result holds for the derived words δ_j.

We study profinite groups in which all commutators are covered by finitely many procyclic subgroups.

Definition

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NO

There is a profinite group in which commutators can be covered by finitely many procyclic groups but G' is infnite not procyclic.

Example

Let A be a finite group such that $A' = \langle a_1 \rangle \times \langle a_2 \rangle$ is an elementary abelian p-group for some prime p.
For each prime $q \neq p$ let B_q be a finite group such that $B_q' = \langle b_q \rangle$ is cyclic of order q.
Let

$$G = A \times \prod_{q \neq p} B_q.$$

Then in every finite quotient of G the derived subgroup is either cyclic or of the form $\langle a_1 \rangle \times \langle a_2 b_{q_1} \dots b_{q_r} \rangle$. Let $b = \prod_{q \neq p} b_q$; then

$$G' \subseteq \langle a_1 a_2 b \rangle \cup \dots \cup \langle a_1^{p-1} a_2 b \rangle \cup \langle a_2 b \rangle \cup \langle a_1 b \rangle,$$

but G' is infinite and not procyclic.

Theorem

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The core of the proof is the following result on finite groups:

Theorem

Let G be a finite group that possesses m cyclic subgroups whose union contains all commutators of G. Then G' has a characteristic subgroup M such that $|M|$ is m-bounded and G'/M is cyclic.

Idea of the proof. We need:

Theorem (Acciarri-Shumyatsky)

Let G be a finite group that possesses m cyclic subgroups whose union contains all coprime commutators of G. Then $\gamma_\infty(G)$ has a subgroup Δ such that

1. Δ is normal in G;
2. $|\Delta|$ is m-bounded;
3. $\gamma_\infty(G)/\Delta$ is cyclic.

Lemma (Acciarri-Shumyatsky)

Let G be a finite non-cyclic p-group that can be covered by m cyclic subgroups. Then $|G|$ is m-bounded.

Theorem (Guralnick)

Let G be a finite p-group such that G' is abelian and generated by at most two elements. Then every element of G' is a commutator.

We factor out suitable subgroups of G of bounded order and we reduce to the case when G' is an abelian p-group generated by at most two elements.

For finite p-groups we can do better:

Theorem

Let p be a prime and let G be a finite p-group in which all commutators can be covered by m cyclic subgroups. Then either G' is cyclic or the order of G' is m-bounded.

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This is the key for studying pro-p-groups:

Theorem

Let p be a prime and let G be a pro-p group such that all commutators in G are covered by m procyclic subgroups. Then either G' is procyclic or it is finite of m-bounded order.

The above results also have a bearing on the case of abstract groups:

Theorem

Let G be a group that possesses m cyclic subgroups whose union contains all commutators of G. Then G' has a characteristic subgroup M such that the order of M is m-bounded and G'/M is cyclic.

Some more results:coverings

The commutators, the lower central words γ_j, the derived words δ_j are all examples of multilinear commutator words, which are words obtained by nesting commutators but using always different variables.
For example the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator word but the Engel word $[x_1, x_2, x_2, x_2]$ is not.
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Definition

The word $w = x_1$ is a multilinear commutator word of weight 1. If u, v are multilinear commutator words of weights m and n respectively involving different variables, then $[u, v]$ is a multilinear commutator word of weight $m + n$.

Theorem (Detomi, Shumyatsky, Morigi)

Let w be a multilinear commutator word and G a profinite group such that G_w can be covered by countably many periodic subgroups. Then $w(G)$ is locally finite.

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Question

Can we say something on profinite groups in which commutators are covered by countably many procyclic subgroups?

Thank you for your attention