Capable special p-groups of rank 2: The isomorphism problem

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Capable special p-groups of rank 2 and exponent p^2

We start with the groups given by the following parameterized presentations:

$$\begin{aligned} G_p(n) = \langle x_1, x_2, y_1, \ldots, y_n \,| x_1^{p^2} &= x_2^{p^2} = y_i^p = [y_i, y_j] = 1, \\ x_1^{x_2} &= x_1^{p+1}, \\ x_1^{y_i} &= x_1^{s_i p+1} x_2^{t_i p}, \\ x_2^{y_i} &= x_1^{u_i p} x_2^{v_i p+1} \rangle \end{aligned} \tag{1}$$

where p is an odd prime, $1 \leq i, j \leq n$, $0 \leq s_i, t_i, u_i, v_i < p$, and $(s_i + v_i) \equiv 0 \pmod{p}$.

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Theorem

The groups $G_p(n)$ defined by (1) are capable.

Goal

The questions to be answered are:

1. Which of these groups are special p-groups of rank 2?

   That is $G' = G^p = Z(G) \cong C_p \times C_p$.
2. What are the isomorphism classes of $G_p(n)$?

Capable special p-groups of rank 2 and exponent p^2

For a fixed prime $p > 2$ and $n > 0$, we associate with each presentation above n matrices over $\mathbb{F}_p$ with trace $0 \pmod{p}$:

$$m_i = \begin{pmatrix} s_i & t_i \\ u_i & -s_i \end{pmatrix} \qquad \text{for } i = 1, \ldots, n.$$

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For some of these matrices their associated presentations are not special p-groups of rank 2. For instance if m_i is the zero matrix for $1 \leq i \leq n$ then

$$G_p(n) \cong C_{p^2} \ltimes C_{p^2} \times \underbrace{C_p \times \cdots \times C_p}_{n}.$$

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Now the subgroup generated by x_1 and x_2 is a 2-generated p-group of class 2 and its isomorphism class is parameterized by the 5-tuple $(2, 1, 1, 1, 0)$ [Ahmad, Magidin, and Morse 2012]. This subgroup is capable [Magidin and Morse 2010].

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Here is a partial result:

Proposition

Let $G \cong K \times C_p$ where $K/Z(K)$ is an elementary abelian p-group. Then if K is capable then G is capable.

Special p-groups of rank 2

Let V be the vector space over $\mathbb{F}_p$ for odd p consisting of the matrices

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Here is the answer to our first question:

Theorem

For $n = 1$, the group $G_p(1)$ is not a special p-group of rank 2 if and only if its associated matrix m_1 has the form

$$\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix} \qquad \textit{for } u \in \mathbb{F}_p.$$

For $n \geq 2$, the group $G_p(n)$ is not a special p-group of rank 2 if and only if for some $1 \leq i, j \leq n$ the associated matrices m_i and m_j are not linearly independent in V.

Special p-groups of rank 2 (cont.)

The vector space V has dimension 3 and hence we obtain the following corollary:

Corollary

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This amounts to partitioning the associated matrices in the vector spaces V, V^2, and V^3 which is induced by the equivalence relation that A and B in V^n are equivalent whenever their associated groups are isomorphic.

The isomorphism classes of $G_p(1)$ (cont.)

We first partition the p^3 matrices in V into those with determinant zero and those with non-zero determinant.

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There are p^2 matrices with determinant zero. From the theorem there are p matrices all with determinant zero that do not define special p-groups of rank 2: $\begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix}$ for all $u \in \mathbb{F}_p$. These matrices are partitioned into two blocks: $u = 0$ and $u \neq 0$ representing two isomorphism classes.

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The other $p^2 - p$ matrices with zero determinant are all associated with isomorphic special p-groups of rank 2, exponent p^2 and order p^5.

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The other $p^2 - p$ matrices with zero determinant are all associated with isomorphic special p-groups of rank 2, exponent p^2 and order p^5.

Those matrices with non-zero determinants partition into two blocks: those whose determinant is a quadratic residue and those whose determinant is a quadratic nonresidue. All elements in each block are associated with isomorphic groups.

The isomorphism classes of $G_p(1)$

These are the isomorphism classes for the groups of order p^5 with presentation (1).

Theorem

The vectors of V are partitioned into the following blocks:

$$\left\{ \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ -r & 0 \end{pmatrix}\right] \right\}$$

where r is a primitive root of $\mathbb{F}_p$. The elements of each block are associated with isomorphic p-groups of exponent p^2 and order p^5. Moreover, the blocks above define pairwise non-isomorphic groups.

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The last three blocks are associated with special p-groups of rank 2.

The isomorphism classes of $G_p(2)$

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The following theorem partitions the elements (A, B) of V^2 where A and B are linearly independent V into three blocks. Each associated with isomorphic special p-groups of rank 2.

The isomorphism classes of $G_p(2)$ (cont.)

Theorem

Consider the set $P_{i,j} = \{(xM_i^Z + yM_j^Z,\ wM_i^Z + zM_j^Z)\}$ for all linearly independent vectors $\{(x,y),(w,z)\}$ in $\mathbb{F}_p \times \mathbb{F}_p$ and all Z in $GL(2,\mathbb{F}_p)$ and fixed elements M_i and M_j in V. Then the elements $(A,B) \in V^2$ such that A and B are linearly independent in V are partitioned into three blocks $P_{1,2}$, $P_{3,4}$, and $P_{5,6}$ corresponding to the following values of M_i and M_j:

$$(M_1, M_2) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) \tag{2}$$

$$(M_3, M_4) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{3}$$

$$(M_5, M_6) = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \right) \tag{4}$$

where r is a primitive root of $\mathbb{F}_p$.

The isomorphism classes of $G_p(2)$ (cont.)

Theorem

All of the elements within $P_{1,2}$, $P_{3,4}$, and $P_{5,6}$ are associated with isomorphic special p-groups of rank 2, exponent p^2 and order p^6, and each block defines groups that are pairwise nonisomorphic.

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All of the elements within $P_{1,2}$, $P_{3,4}$, and $P_{5,6}$ are associated with isomorphic special p-groups of rank 2, exponent p^2 and order p^6, and each block defines groups that are pairwise nonisomorphic.

Corollary

Taking the basis $((1,0),(0,1))$ of $\mathbb{F}_3 \times \mathbb{F}_3$ we obtain the following representatives for the isomorphism classes for the capable special p-groups of rank 2 of exponent p^2 and order p^6:

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right], \left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right], \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}\right]$$

where r is a primitive root of $\mathbb{F}_p$.

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This reduces to partitioning V^3 into all (A, B, C) such that A, B, and C are linearly dependent and those that are linearly independent in V. This partitions V^3 into those groups that are not special p-groups of rank 2 and those that are not.

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This reduces to partitioning V^3 into all (A, B, C) such that A, B, and C are linearly dependent and those that are linearly independent in V. This partitions V^3 into those groups that are not special p-groups of rank 2 and those that are not.

Theorem

All (A, B, C) such that A, B, and C are linearly independent in V define isomorphic special p-groups of rank 2, exponent p^2 and of order p^7. Hence there is one equivalence class with all groups isomorphic. One representative is

$$\left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right].$$