Galois pro-p groups with constant generating numbers

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Profinite groups
and pro-p groups

Let p be a prime. (Usually $p \neq 2$...) A Hausdorff topological group is called pro-p if it is compact and 1 has a basis of neighbourhood consisting of clopen normal subgroups of index a p-power:

$$G = \varprojlim_{i \in I} G_i, \quad \text{with } |G_i| = p^n \; \forall\, i \in I.$$

Example

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} = \left\{ a_0 + a_1 p + a_2 p^2 + \ldots, a_i \in \mathbb{F}_p \right\}$$

Note that here $p^n \to 0$ for $n \to \infty$.

For G (topologically) finitely generated

$$d(G) := \text{minimal } \sharp \text{ of topological generators of } G$$

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Iwasawa's question
for pro-p groups

In the '80s Iwasawa posed the following question: fixed a positive n, which finitely generated pro-p groups satisfy the formula

$$d(U) - n = |G : U|(d(G) - n) \quad \forall\, U \leq_o G \, ? \tag{1}$$

(He observed they have interesting representation-theoretic properties.)

If $n = 1$ then (1) is the (topological) Schreier formula, thus G is a free pro-p group. If $n = d(G)$ then (1) becomes

$$d(U) = d(G) \quad \forall\, U \leq_o G \tag{2}$$

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Locally powerful groups

"almost" abelian

A (pro-)p group G is called powerful if $[G,G] \leq \begin{cases} G^p & \text{for } p \neq 2 \\ G^4 & \text{for } p = 2 \end{cases}$

Namely the commutators are "close" enough to 1.

A pro-p group G is called locally powerful if every closed finitely generated subgroup is powerful.

Locally powerful pro-p groups...

which are torsion-free are precisely the pro-p groups with a presentation

$$G = \left\langle \sigma, \tau_1, \ldots, \tau_n \,\middle|\, [\tau_i, \tau_j] = 1, [\sigma, \tau_i] = \tau_i^{p^k}\ \forall i, j,\ k \in \mathbb{N}^* \cup \{\infty\} \right\rangle$$

We can write G as the semi-direct product $G = \mathbb{Z}_p \ltimes A$, $A \simeq \mathbb{Z}_p^m$, the action being induced by the multiplication by $1 + p^k$.

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Absolute Galois groups satisfying (2)

are locally powerful

The absolute Galois group $G_{\mathbf{K}}$ of a field $\mathbf{K}$ is the Galois group of the separable algebraic closure:

$$G_{\mathbf{K}} := \mathrm{Gal}\left(\bar{\mathbf{K}}^{sep}/\mathbf{K}\right).$$

We ask which $G_{\mathbf{K}}$'s which are finitely generated pro-p satisfy (2)...

Theorem

Let G be a finitely generated pro-p group realizable as absolute Galois group of a field. Then (2) holds if, and only if, G is locally powerful.

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Number of defining relations
for Absolute Galois pro-p groups

This kind of absolute Galois pro-p groups have the highest number of defining relations. Indeed, for a minimal presentation $1 \to R \to F \to G \to 0$ one has the following bounds:

$$0 \leq \sharp \text{ of defining relations of } G \leq \binom{d(G)}{2}$$

(The left-hand side are free pro-p groups, the right-hand side are locally powerful pro-p groups.) In particular...

Theorem

Let G be an absolute Galois pro-p group. Then either G is locally powerful, or it contains a free closed (non-abelian) pro-p group.

Last slide
Thanks & main references

THANK YOU!

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