Group Algebras with the Bounded Splitting Property

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A classical result on abelian groups

Let A be an abelian group with torsion subgroup T. In general T is not a direct summand of A.

For example, let

$$A = \prod_p \mathbb{Z}_p;$$

then

$$T = \sum_p \mathbb{Z}_p$$

and A/T is an uncountable torsion-free divisible group. Since A is residually finite, T cannot be a direct summand.

In 1936-7 R. Baer and S. Fomin proved the following result.

Theorem. *If T is bounded, i.e., of finite exponent, then it is a direct summand of A*

Question: How can this be generalized to modules over rings?

Some definitions

Let R be a ring (with identity element). A left ideal L of R is called *essential* if $L \cap N \neq 0$ for every non-zero left ideal N of R. In a domain every non-zero ideal is essential.

Let M be a left R-module. The *singular submodule*

$$Z(M)$$

is the subset of all $a \in M$ such that $\mathrm{Ann}_R(a)$ is an essential left ideal of R. If R is a domain, then $Z(M)$ is just the torsion submodule of M

The bounded splitting property for rings

The ring R has the the *bounded splitting property* if, for any left R-module M,

$$Z(M) \text{ is a direct summand of } M$$

whenever $Z(M)$ has *bounded order*, i.e., it embeds in a left R-module generated by elements which are annihilated by a fixed essential left ideal.

Some examples of rings with the BSP

1. *Every principal ideal domain has the BSP*.

For, let M be an R-module; then $Z(M) = T$, the torsion submodule. If T has bounded order, it embeds in a module which is annihilated by some $r \neq 0 \in R$. Then $r \cdot T = 0$ and this is known to imply that T is a direct summand of M, cf. the case $R = \mathbb{Z}$.

2. *A semisimple ring has the BSP.*

This is because every R-module is semisimple and hence every submodule is a direct summand.

Problem. Let G be a group and K a field. Find necessary and sufficent conditions on (G, K) for the group algebra $K[G]$ to have the BSP.

When G is finite, there is a simple answer.

Theorem 1 (M. Teply and B. Torrecillas 1997). *Let G be a finite group and K a field. Then $K[G]$ has the BSP if and only if it is a semisimple ring, i.e., $\mathrm{char}(K)$ does not divide $|G|$.*

Sketch of proof. Only the sufficiency is in doubt. Assume that $R = K[G]$ has the BSP. Then $K[G]$ is a Frobenius ring, which implies that $Z(K[G]) = \mathrm{Jac}(K[G]) = J$. Since G is finite, $Z(K[G])$ is bounded and the BSP implies that J is a direct summand of $K[G]$. Hence $R = J \oplus L$ with L a left ideal. But $J = \mathrm{Jac}(K[G])$, so $R = L$ and $J = 0$ and R is semisimple. $\square$

A necessary condition for the BSP

Lemma 1. (Teply and Torrecillas) *Let G be a group and K a field. If $K[G]$ has the BSP, then every subnormal subgroup of G is finite or of finite index, [Property $\mathcal{S}$].*

This raises the question: which groups have property $\mathcal{S}$?

Clearly all simple groups do, but for groups with some weak form of solubility, $\mathcal{S}$ has strong consequences.

Theorem 2. *Let G be a group with an ascending series whose factors are finite or locally nilpotent. Then G has $\mathcal{S}$ if and only if it is cyclic-by-finite or quasicyclic-by-finite.*

This includes the case of hyperfinite groups, soluble groups and more generally radical groups.

Proof of the soluble case.

Proof. Let G be an infinite soluble group with $\mathcal{S}$. Then G has an infinite abelian subnormal subgroup A and $|G : A|$ is finite. If A contains an element a of infinite order, then $\langle a \rangle$ is subnormal in G and $|G : \langle a \rangle|$ is finite, so G is cyclic-by-finite. Assume A is periodic. It cannot contain an infinite direct product of non-trivial groups. Hence A has finite total rank. From the structure of such groups A is a quasicyclic group and G is quasicyclic-by-finite. $\square$

Fields of the first kind

In 1997 Teply and Torrecillas gave a characterization of the infinite abelian groups A and fields K such that $K[A]$ has the BSP.

This involves a special type of field. A field K is *of the first kind* with respect to a prime p if $p \neq \mathrm{char}(K)$ and $K(\zeta_2) \neq K(\zeta_i)$ for some $i > 2$ where ζ_i is a primitive p^ith root of unity in the algebraic closure of K. Otherwise K is a *field of the second kind*.

For example prime fields and algebraic number fields are of the first kind: algebraically closed fields are of the second kind.

The principal theorem

Theorem 3. *Let K be a field of characteristic $q \geq 0$ and let G be an infinite group with an ascending series with finite or locally nilpotent factors. Then $K[G]$ has the BSP if and only if one of the following is true.*

1. There is subgroup C of type p^∞ with finite index and K is a field of the first kind with respect to p.

2. There is a finite normal subgroup F such that $q \notin \pi(F)$ and either (a) $G/F \simeq \mathbb{Z}$ or (b) $G/F \simeq \mathrm{Dih}(\infty)$ and $q \neq 2$.

Sketch of proof – necessity

Assume $K[G]$ has the BSP. Then since G has the property $\mathcal{S}$, Theorem 1 shows that there exists C normal in G, with C in finite cyclic or quasicyclic and G/C finite.

For example, consider the case $C \simeq \mathbb{Z}$. Suppose that $C \leq Z(G)$; then G' is finite and there exists a characteristic finite subgroup F such that $G/F \simeq \mathbb{Z}$. Since $K[G]$ has the BSP, so does $K[F]$ and $q \notin \pi(F)$ by Theorem 1. Thus we are in situation 2(a).

Next assume that $C \not\leq Z(G)$ and let $D = C_G(C)$. Then $|G : D| = 2$ and $G = \langle g, D \rangle$. As before there is a finite characteristic subgroup F of D such that $D/F \simeq \mathbb{Z}$. Also F is normal in G and g inverts elements of D/F. Thus $G/F \simeq \mathrm{Dih}(\infty)$.

Sketch of proof – necessity

Suppose that $q = 2$. Since KF has the BSP, $|F|$ is odd and we can choose $|g| = 2$. Hence $G = \langle g \rangle \ltimes D$, so $K[G] = (KD) * \langle g \rangle$, the skew group ring. This is not possible since $\mathrm{char}(K) = 2 = |g|$. Hence $q > 2$ and we are in situation 2(b).

Sketch of proof – sufficiency

First note the following result.

Lemma 2 (Teply and Torrecillas). *Let H a finite group and R be a strongly graded ring of type H where $|H|$ is invertible in R. Then R has the BSP if and only if R_1 has the BSP.*

Assume that sufficiency in Theorem 3 has been proved in case 2(a). Now suppose that condition 2(b) holds. Then there exists a finite $F \lhd G$ such that $D/F \simeq \mathbb{Z}$ and $|G : D| = 2$.

Sketch of proof – sufficiency

Since $q \notin \pi(F)$, the sufficiency of condition 2(a) implies that $K[D]$ has the BSP. Also $K[G] = (KD) * (G/D)$ and q does not divide $|G/D|$ since $q \neq 2$. Apply Lemma 2 with $R = K[G]$, $H = G/D$ and $R_1 = KD$. Therefore $K[G]$ has the BSP.

An open problem

Problem

Let G be an infinite simple locally finite group. Does there exist a field K such that $K[G]$ has the BSP?