“If you search, you will find something.”

David, summer 2011.

Ischia Group Theory 2014

The unitary cover of a finite group

and the exponent of the Schur multiplier

Nicola Sambonet - Technion, Haifa Israel

Acknowledgment.

I will present the advancement of my doctoral thesis, I would like to thank my advisors:

- Prof. Eli Aljadeff, Technion, Haifa, Israel
- Dr. Yuval Ginosar, University of Haifa, Israel

Introduction: on the exponent of M(G)

The "**Schur mulltiplier**" of a finite group G is: $\mathrm{M}(G) = \mathrm{H}^2(G, \mathbb{C}^\times)$

- projective representations
- central extensions

Introduction: on the exponent of M(G)

The "**Schur mulltiplier**" of a finite group G is: $\mathrm{M}(G) = \mathrm{H}^2(G, \mathbb{C}^\times)$

• projective representations • central extensions

Schur (1904): $[\exp \mathrm{M}(G)]^2 \mid |G|$

Introduction: on the exponent of M(G)

The "**Schur mulltiplier**" of a finite group G is: $\mathrm{M}(G) = \mathrm{H}^2(G, \mathbb{C}^\times)$

- projective representations
- central extensions

Schur (1904): $[\exp \mathrm{M}(G)]^2 \mid |G|$

e.g. $\exp \mathrm{M}(C_n \times C_n) = n$

Problem: which groups satisfy the following condition?

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

Introduction: on the exponent of M(G)

The "**Schur mulltiplier**" of a finite group G is: $\mathrm{M}(G) = \mathrm{H}^2(G, \mathbb{C}^\times)$

• projective representations • central extensions

Schur (1904): $[\exp \mathrm{M}(G)]^2 \mid |G|$

e.g. $\exp \mathrm{M}(C_n \times C_n) = n$

Problem: which groups satisfy the following condition?

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

Abelian groups: for $A = C_{d_1} \oplus \cdots \oplus C_{d_n}$, $d_i \mid d_{i+1}$, it is known that

$$\mathrm{M}(A) = (C_{d_1})^{n-1} \oplus \cdots \oplus (C_{d_i})^{n-i} \oplus \cdots \oplus C_{d_{n-1}}$$

consequently $\exp \mathrm{M}(A) = d_{n-1} \mid \exp A = d_n$, and A satisfies $\circledast$.

Introduction: on the exponent of M(G)

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

- abelian groups
- finite simple groups (FSGC)
- many others...

Introduction: on the exponent of M(G)

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

• abelian groups • finite simple groups (FSGC) • many others...

Reduction to p-groups: $\mathrm{M}(G)_p$ is embedded in $M(G_p)$, therefore

$$\exp \mathrm{M}(G_p) \mid \exp G_p \ \ \forall p \ \Rightarrow \ \exp \mathrm{M}(G) \mid \exp G\,.$$

Introduction: on the exponent of M(G)

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

- abelian groups
- finite simple groups (FSGC)
- many others...
- powerful p-groups ($p > 2$, $\mho(G) \leq G'$) [A. Lubotzky and A. Mann (1987)]
- $p > 2$ and nilpotency class at most 4,
  metabelian groups of prime exponent, ... [P. Moravec (2006,"08,"11)]

Reduction to p-groups: $\mathrm{M}(G)_p$ is embedded in $M(G_p)$, therefore

$$\exp \mathrm{M}(G_p) \mid \exp G_p \;\; \forall p \;\Rightarrow\; \exp \mathrm{M}(G) \mid \exp G\ .$$

Introduction: on the exponent of M(G)

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

• abelian groups • finite simple groups (FSGC) • many others...

• powerful p-groups ($p > 2$, $\mho(G) \leq G'$) [A. Lubotzky and A. Mann (1987)]

• $p > 2$ and nilpotency class at most 4,
metabelian groups of prime exponent, ... [P. Moravec (2006,"08,"11)]

Reduction to p-groups: $\mathrm{M}(G)_p$ is embedded in $M(G_p)$, therefore

$$\exp \mathrm{M}(G_p) \mid \exp G_p \ \ \forall p \ \Rightarrow \ \exp \mathrm{M}(G) \mid \exp G \, .$$

A. J. Bayes, J. Kautsky and J. W. Wamsley (1973): $\circledast$ does not hold for all 2-groups

Introduction: on the exponent of M(G)

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

• abelian groups • finite simple groups (FSGC) • many others...

• powerful p-groups ($p > 2$, $\mho(G) \leq G'$) [A. Lubotzky and A. Mann (1987)]

• $p > 2$ and nilpotency class at most 4,
metabelian groups of prime exponent, ... [P. Moravec (2006,"08,"11)]

Reduction to p-groups: $\mathrm{M}(G)_p$ is embedded in $M(G_p)$, therefore

$$\exp \mathrm{M}(G_p) \mid \exp G_p \ \ \forall p \ \Rightarrow \ \exp \mathrm{M}(G) \mid \exp G \, .$$

A. J. Bayes, J. Kautsky and J. W. Wamsley (1973): $\circledast$ does not hold for all 2-groups

• it is still an open problem for groups of odd order, or at least of prime exponent

• it is of interest to provide other bounds

For instance, this has been done by Lubotzky and Mann, and by Moravec.

Introduction: central extensions

A "central extension" of G is an exact sequence of groups

$$1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \ , \quad A \leq Z(\Gamma) \ .$$

Introduction: central extensions

A "central extension" of G is an exact sequence of groups

$$1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \ , \ A \leq Z(\Gamma) \ .$$

- Let $\phi : G \to \Gamma$ be a section (i.e. $\pi \circ \phi(g) = g$), this defines

$$\alpha_\phi : G \times G \to A \ , \ \phi(g) \cdot \phi(h) = \alpha_\phi(g, h) \cdot \phi(gh) \ .$$

Introduction: central extensions

A "central extension" of G is an exact sequence of groups

$$1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \ , \ \ A \leq Z(\Gamma) \ .$$

- Let $\phi : G \to \Gamma$ be a section (i.e. $\pi \circ \phi(g) = g$), this defines

$$\alpha_\phi : G \times G \to A \ \ , \ \ \phi(g) \cdot \phi(h) = \alpha_\phi(g,h) \cdot \phi(gh) \ .$$

- $Z^2(G,A) = \{ \, \alpha_\phi \, , \text{ for some } (\Gamma, \phi) \, \}$ "cocycles"
- $B^2(G,A) = \{ \ \text{"} \ \ \text{with } \Gamma = A \times G \, \}$ "coboundaries"

Introduction: central extensions

A "central extension" of G is an exact sequence of groups

$$1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \ , \quad A \leq Z(\Gamma) \ .$$

• Let $\phi : G \to \Gamma$ be a section (i.e. $\pi \circ \phi(g) = g$), this defines

$$\alpha_\phi : G \times G \to A \ , \quad \phi(g) \cdot \phi(h) = \alpha_\phi(g,h) \cdot \phi(gh) \ .$$

• $Z^2(G,A) = \{ \alpha_\phi \ , \text{ for some } (\Gamma, \phi) \}$ "cocycles"

• $B^2(G,A) = \{$ " with $\Gamma = A \times G \}$ "coboundaries"

• $H^2(G,A) = Z^2(G,A)/B^2(G,A)$ "2nd-cohomology group"

($\mathrm{M}(G) = H^2(G, \mathbb{C}^\times)$ "Schur multiplier").

Introduction: central extensions

A "central extension" of G is an exact sequence of groups

$$1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \ , \ \ A \leq Z(\Gamma) \ .$$

- Let $\phi : G \to \Gamma$ be a section (i.e. $\pi \circ \phi(g) = g$), this defines

$$\alpha_\phi : G \times G \to A \ , \ \ \phi(g) \cdot \phi(h) = \alpha_\phi(g,h) \cdot \phi(gh) \ .$$

- $Z^2(G,A) = \{ \alpha_\phi \ , \text{ for some } (\Gamma,\phi) \}$ "cocycles"
- $B^2(G,A) = \{ \ \text{"} \ \text{ with } \Gamma = A \times G \}$ "coboundaries"
- $H^2(G,A) = Z^2(G,A)/B^2(G,A)$ "2nd-cohomology group"

($\mathrm{M}(G) = H^2(G, \mathbb{C}^\times)$ "Schur multiplier").

- Given Γ, it is defined "the standard map"

$$\eta : \mathrm{Hom}(A, \mathbb{C}^\times) \to \mathrm{M}(G) \ , \ \ \lambda \mapsto [\lambda \circ \alpha_\phi] \ .$$

Then, Γ has the "**projective lifting property**" if η is onto.

Schur's theorem & the untiary cover

There exists a "**Schur cover**", i.e. a central extension

$$1 \to A \to \Gamma \to G \to 1 \ , \ \ A \leq Z(\Gamma)$$

which has the projective lifting property, and such that $A \simeq \mathrm{M}(G)$.

Schur's theorem & the untiary cover

There exists a "**Schur cover**", i.e. a central extension

$$1 \to A \to \Gamma \to G \to 1 \ \ , \ \ A \leq Z(\Gamma)$$

which has the projective lifting property, and such that $A \simeq \mathrm{M}(G)$.

Sketch of Proof. $B^2(G, \mathbb{C}^\times)$ has a complement J in $Z^2(G, \mathbb{C}^\times)$, from this Schur constructed the extension Γ satisfying the assertion.

Schur's theorem & the untiary cover

There exists a "**Schur cover**", i.e. a central extension

$$1 \to A \to \Gamma \to G \to 1 \ , \ \ A \leq Z(\Gamma)$$

which has the projective lifting property, and such that $A \simeq \mathrm{M}(G)$.

Sketch of Proof. $B^2(G, \mathbb{C}^\times)$ has a complement J in $Z^2(G, \mathbb{C}^\times)$, from this Schur constructed the extension Γ satisfying the assertion.

• We mimic Schur's construction, replacing J with the subgroup

$$Z_{\mathrm{u}}(G, \mathbb{C}^\times) = \{ \, \alpha_\phi \in Z^2(G, \mathbb{C}^\times) \ \ , \ \ \phi(g)^{o(g)} = 1 \}$$

Schur's theorem & the untiary cover

There exists a "**Schur cover**", i.e. a central extension

$$1 \to A \to \Gamma \to G \to 1 \;\;,\;\; A \leq Z(\Gamma)$$

which has the projective lifting property, and such that $A \simeq \mathrm{M}(G)$.

Sketch of Proof. $B^2(G, \mathbb{C}^\times)$ has a complement J in $Z^2(G, \mathbb{C}^\times)$, from this Schur constructed the extension Γ satisfying the assertion.

• We mimic Schur's construction, replacing J with the subgroup

$$Z_{\mathrm{u}}(G, \mathbb{C}^\times) = \{\, \alpha_\phi \in Z^2(G, \mathbb{C}^\times) \;\;,\;\; \phi(g)^{o(g)} = 1\}$$

There exists a canonical central extension, the "**unitary cover**" of G,

$$1 \to A \to \Gamma_{\mathrm{u}}(G) \to G \to 1 \;\;,\;\; A \leq Z(\Gamma_{\mathrm{u}}(G))$$

which has minimal exponent satisfying the projective lifting property.

As we searched...

Main result. For any normal subgroup N of G:

- $\exp \mathrm{M}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \mathrm{M}(G/N)$
- $\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \Gamma_{\mathrm{u}}(G/N)$
- $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

As we searched...

Main result. For any normal subgroup N of G:

- $\exp \mathrm{M}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \mathrm{M}(G/N)$
- $\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \Gamma_{\mathrm{u}}(G/N)$
- $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

$\rightarrow$ we obtain new bounds for subnormal series, which improve those previously known.

As we searched...

Main result. For any normal subgroup N of G:

- $\exp \mathrm{M}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \mathrm{M}(G/N)$
- $\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \Gamma_{\mathrm{u}}(G/N)$
- $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

$\rightarrow$ we obtain new bounds for subnormal series, which improve those previously known.

e.g. if G is powerful (or else in case $p = 2$), then $\exp \Gamma_{\mathrm{u}}(G) = \exp G$.

If G is a regular p-group and $\exp \mathrm{M}(G/\mho(G))$ divides p, then

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

As we searched...

Main result. For any normal subgroup N of G:

- $\exp \mathrm{M}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \mathrm{M}(G/N)$
- $\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(N) \cdot \exp \Gamma_{\mathrm{u}}(G/N)$
- $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

$\rightarrow$ we obtain new bounds for subnormal series, which improve those previously known.

e.g. if G is powerful (or else in case $p = 2$), then $\exp \Gamma_{\mathrm{u}}(G) = \exp G$.

If G is a regular p-group and $\exp \mathrm{M}(G/\mho(G))$ divides p, then

$$\exp \mathrm{M}(G) \mid \exp G \qquad \circledast$$

- $\circledast$ holds for regular p-groups iff it holds for groups of exponent p.
- absolutely regular p-groups enjoy $\circledast$, and in general regular 3-groups.

Zel'manov solution of the RBP

Recall: $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

Let $\mathfrak{S}(G)$ denote the set of 2-generator subgroups of G, then

$$\exp \Gamma_{\mathrm{u}}(G) \mid \operatorname*{lcm}_{S \in \mathfrak{S}(G)} \exp \Gamma_{\mathrm{u}}(S) \, .$$

Zel'manov solution of the RBP

Recall: $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

Let $\mathfrak{S}(G)$ denote the set of 2-generator subgroups of G, then

$$\exp \Gamma_{\mathrm{u}}(G) \mid \operatorname*{lcm}_{S \in \mathfrak{S}(G)} \exp \Gamma_{\mathrm{u}}(S) .$$

By the Zel'manov solution of the Restricted Burnside Problem, there exists a finite group

$$\mathfrak{B}_{p^k} = \mathrm{RBP}(2, p^k)$$

such that every element in $\mathfrak{S}(p^k)$ is a homomorphic image of $\mathfrak{B}_{p^k}$.

Zel'manov solution of the RBP

Recall: $\Gamma_{\mathrm{u}}(G/N)$ is a homomorphic image of $\Gamma_{\mathrm{u}}(G)$.

Let $\mathfrak{S}(G)$ denote the set of 2-generator subgroups of G, then

$$\exp \Gamma_{\mathrm{u}}(G) \mid \operatorname*{lcm}_{S \in \mathfrak{S}(G)} \exp \Gamma_{\mathrm{u}}(S) .$$

By the Zel'manov solution of the Restricted Burnside Problem, there exists a finite group

$$\mathfrak{B}_{p^k} = \mathrm{RBP}(2, p^k)$$

such that every element in $\mathfrak{S}(p^k)$ is a homomorphic image of $\mathfrak{B}_{p^k}$.

If G is a group of exponent p^k, then

$$\exp \Gamma_{\mathrm{u}}(G) \mid \exp \Gamma_{\mathrm{u}}(\mathfrak{B}_{p^k}) .$$

Moreover, $\exp \Gamma_{\mathrm{u}}(\mathfrak{B}_{p^k}) = p^k \cdot \exp \mathrm{M}(\mathfrak{B}_{p^k})$ and $p^k \mid \exp \mathrm{M}(\mathfrak{B}_{p^k})$.