# Exponent of locally finite groups with small centralizers

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The proof of the above theorem uses the classification of finite simple groups and it seems unlikely that one could find a proof that does not use the classification.

In particular the classification is used to obtain

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*Let $A$ be a finite $p$-group acting on a finite group $G$ of exponent $e$. Assume $|C_G(A)| \leq m$. Then $|G : S(G)|$ is $\{|A|, e, m\}$-bounded.*

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*Let $G$ be a locally finite group of finite exponent containing a finite $p$-subgroup whose centralizer is finite. Then $G$ is almost locally soluble.*

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The idea of the proof of

**Theorem**

*Let $G$ be a locally finite group containing a non-cyclic subgroup $A$ of order $p^2$ such that $C_G(A)$ is finite and $C_G(a)$ has finite exponent for all $a \in A^\#$. Then $G$ is almost locally soluble and has finite exponent.*

Under the hypothesis of the theorem one can show that the Sylow $p$-subgroups of $G$ are finite. Thus, we can use induction on the order of the Sylow $p$-subgroups of $G$.
It is sufficient to show that $G$ has finite exponent.

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