On groups satisfying cubic conditions

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1 Groups defined by polynomial conditions

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General Context.

Given a free group F_d of rank d , freely generated by $\{y_1, ..., y_d\}$, the group ring $\mathbb{D}F_d$ over an integral domain $\mathbb{D}$ and a set E_d of elements of F_d. Assign to each s in E_d a monic polynomial in one variable $p_s(t)$ over $\mathbb{D}$. Consider the finitely presented ring

$$A = \frac{\mathbb{D}F_d}{\langle p_s(s) \rangle_{ideal}}.$$

This is the stuff of algebraic groups.

Our main interest though is in E_d finite. .

2 Groups with low polynomial degree

- Polynomial degree 2.

Let

$$E_d = \left\{ y_i \ (1 \le i \le d), y_i y_j \ (1 \le i < j \le d) \right\}$$

Then easily, A has $\mathbb{Z}$-rank 2^d.

- Polynomial degree 3. Not so easy.

Proposition. Let $d = 2$, and $E_2 = \left\{ y_1, y_2, y_1 y_2, y_1 y_2^{-1} \right\}$. Then A has $\mathbb{D}$-rank at most 39.

Observe the analogy with the presentation of the Burnside 3-group, $B(2,3)$.

More generally,

Theorem 1. *Define the following subsets of* F_d

$$E_1 = \{y_1\}\,,\ M_1 = \left\{e, y_1^{\pm 1}\right\}$$

and inductively for $1 \leq s \leq d-1$*,*

$$\begin{aligned} E_{s+1} &= E_s \cup M_s y_{s+1}^{\pm 1}, \\ M_{s+1} &= M_s \cup M_s y_{s+1}^{\pm 1} M_s \\ &\quad \cup\ M_s y_{s+1}^{-1} M_s^{\#} y_{s+1} M_s. \end{aligned}$$

Then M_d generates A over $\mathbb{D}$ and $\log_3 |M_d| \leq 4.3^{d-2}$.

Question. How precise are the bounds?

- Degree 4. **Unknown!**

3 Unipotent conditions

Suppose all elements of a group G satisfy $(x-1)^n = 0$. This is the realm of variety of representations over fields.

- (Kolchin) If G is a subgroup of a finite dimensional algebra then G is nilpotent.

- (Zelmanov) If the characteristic of the field is zero, or 'very large' compared to n, then G is nilpotent.

Back to small degrees.

3.1 Unipotent quadratic conditions.

Here we can easily obtain maximum information.

Let the image of F_d in A be G and the image of y_i be a_i.

Theorem2. Define the quotient ring $A_d(2) = \frac{\mathbb{Z}F_d}{\left\langle (x-1)^2 | x \in E_d \right\rangle}$ where

$$E_d = \left\{ y_i \ \ (1 \leq i \leq d), y_i y_j \ \ (1 \leq i < j \leq d) \right\}$$

and B_d is the augmentation ideal $\omega(A_d(2))$.

Then: B_d is commutative, nilpotent of degree $d+1$, has torsion subgroup $\mathrm{Tor}(B_d) = B_d^d = \mathbb{Z}.(a_1 ... a_d - 1)$ and $2.B_d^d = 0$.

Also, G_d is a free d-generated nilpotent group of class 2.

3.2 Unipotent cubic conditions.

The situation here is not as easy.

Let $\mathbb{D} = \mathbb{Z}\left[\frac{1}{6}\right]$.

Theorem 3. Define the quotient ring $A_N = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 | x \in N \right\rangle}$,
where

$$N = \left\{ y_1, y_2, y_1 y_2, y_1^{-1} y_2, y_1^2 y_2, y_1 y_2^2, [y_1, y_2] \right\}$$

and let G be the image of F_2 in A_N. Then
(i) rank $A_N = 18$,
(ii) $\omega(A_N)$ is nilpotent of degree 6,
(iii) G is a free 2-generated nilpotent group of degree 5,
(iv) $(x-1)^3 = 0$ is satisfied by all elements of G.

In general, for $d \geq 2$,

Theorem 4. There exists a finite subset P_d of F_d such that the quotient ring $A_d(3) = \frac{\mathbb{D}F_d}{\left\langle (x-1)^3 | x \in P_d \right\rangle}$ has finite $\mathbb{D}$-rank and the augmentation ideal $\omega(A_d(3))$ is nilpotent.

3.3 Analysis of conditions

We introduce cubic conditions in steps.

- Step 1. Define the ring $A_Q = \frac{\mathbb{Z}F_2}{\left\langle (x-1)^3 | x \in Q \right\rangle}$ where

$$Q = \{y_1, y_2, y_1 y_2\}.$$

One quotient of A_Q is the group algebra over $GF(3)$ of the infinite Euclidean triangle group

$$\left\langle y_1, y_2 \mid y_1^3 = y_2^3 = (y_1 y_2)^3 = e \right\rangle.$$

Then ring A_Q is freely generated as a $\mathbb{Z}$-module by 1 and monomials in U, V which avoid having subwords from

$$\left\{ (UV)^2, (VU)^2, U^2V^2U^2, V^2U^2V^2 \right\}.$$

Question. What are quotients of A_Q?

- Step 2. Define the ring $A_S = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 | x \in S \right\rangle}$ where

  $$S = \left\{ y_1, y_2, y_1 y_2, y_1^{-1} y_2 \right\}.$$

  Then A_S has $\mathbb{D}$-rank 23 and $\omega(A_S)$ is nilpotent of degree 7.

Any 3-dimensional representation of A_S such that

$$\begin{aligned} a_1 &\rightarrow \alpha = \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix}, a_{21} a_{32} \neq 0, \\ a_2 &\rightarrow \beta = \left(b_{ij} \right)_{3 \times 3} \end{aligned}$$

is lower uni-triangular (with the use of Maple).

- Step 3. Define the ring $A_T = \frac{\mathbb{D}F_2}{\left\langle (x-1)^3 | x \in T \right\rangle}$ where

$$T = \left\{ y_1, y_2, y_1 y_2, y_1^{-1} y_2, y_1^2 y_2, y_1 y_2^2 \right\}.$$

Then

$$\begin{aligned} & A_T \text{ has } \mathbb{D}\text{-rank } 19, \\ \omega\left(A_T\right)^6 &= \mathbb{D}.VU^2V^2U \\ \omega\left(A_T\right)^7 &= 0. \end{aligned}$$

3.4 Handcraft Computations

Let a and b be the images of y_1, y_2 in each the quotient rings above. Also, let $U = a - 1, V = b - 1$.We start with the relations $(a-1)^3 = (b-1)^3 = (ab-1)^3 = 0$ which translate to $U^{\;3} = V^3 = 0, (UV + V + U)^3 = 0$.

Order $U < V < U^2 < V^2$. The substitutions are chosen in the following order: decrease the syllable length of monomials, decrease the total length, decrease total V length, decrease using reverse lexicographical ordering. The following computations were obtained by hand and checked with GAP.

(1):=

$$
\begin{aligned}
& (UV)^3 \\
= \; & -VUVUV - UVUVU - UVUV^2 - UV^2UV \\
& -UVU^2V - U^2VUV \\
& -2UVUV - VUVU - VUV^2 - V^2UV - VU^2V \\
& -UV^2U - UVU^2 - U^2VU - VUV - UVU \\
& -U^2V^2 - UV^2 - V^2U - U^2V - VU^2;
\end{aligned}
$$

(2):=

$$
\begin{aligned}
& UVUVU \\
= \ & -UVUV - VUVU - UV^2U - UVU^2 - U^2VU \\
& -VUV - UVU - UV^2 - V^2U - U^2V - VU^2;
\end{aligned}
$$

(3):=

$$
\begin{aligned}
& VUVUV \\
= \ & -UVUV - VUVU - VUV^2 - V^2UV - VU^2V \\
& -VUV - UVU - UV^2 - V^2U - U^2V - VU^2;
\end{aligned}
$$

(4):=

$$
\begin{aligned}
& UVUV \\
= \ & -VUV - UVU + V^2U^2 - UV^2 - V^2U \\
& -U^2V - VU^2;
\end{aligned}
$$

(5):=

$$
\begin{aligned}
VUVU &= -VUV - UVU + U^2V^2 - UV^2 \\
&\quad -V^2U - U^2V - VU^2;
\end{aligned}
$$

(6):=

$$
\begin{aligned}
U^2V^2U^2 &= UV^2U^2 + U^2V^2U + U^2VU^2 - UV^2U \\
&\quad -UVU^2 - U^2VU + VUV + UVU \\
&\quad -U^2V^2 - V^2U^2 + UV^2 + V^2U \\
&\quad +U^2V + VU^2;
\end{aligned}
$$

(7):=

$$
\begin{aligned}
& V^2U^2V^2 \\
= \ & V^2UV^2 + VU^2V^2 + V^2U^2V - VUV^2 - V^2UV \\
& -VU^2V + VUV + UVU - U^2V^2 - V^2U^2 \\
& +UV^2 + V^2U + U^2V + VU^2.
\end{aligned}
$$

Now, add the relation $(a^{-1}b - 1)^3 = 0$. *This has the consequence* $(ab^{-1} - 1)^3 = 0$. *We obtain new relaions by making substitution*

$$a \mapsto a^{-1}, b \mapsto \ b$$

which *induces*

$$U \mapsto U^2 - U, V \mapsto V$$

in the augmentation ideal. Then

(8):=

$$\begin{aligned} & VU^2V \\ = \ & UV^2U + VUV - UVU + UV^2 + V^2U \\ & -U^2V - VU^2; \end{aligned}$$

(9):=

$$\begin{aligned} V^2UV^2 \ = \ & U^2VU^2 + VUV^2 + V^2UV \\ & -UVU^2 - U^2VU - VUV + UVU \\ & -UV^2 - V^2U + U^2V + VU^2; \end{aligned}$$

(10):=

$$U^2VU^2V = VU^2VU^2;$$

At the 38th step we obtain

$$3VU^2V^2 = -3V^2U^2V+3UV^2U^2+3U^2V^2U+6VUV^2+6V^2UV-6UVU^2-6U^2VU;$$

- This is the first instance where torsion appears.

The computation continues for 53 steps, under the assumption that 6 is invertible, and finally produces the following table of basic relations in the augmentation ideal $\omega(A)$ with the symmetry $U \leftrightarrow V$:

$$VUV = UVU - UV^2 + U^2V - V^2U + VU^2, \quad (1)$$

$$UV^2U = 2\left(UVU + U^2V + VU^2\right) - U^2V^2 - V^2U^2, \quad (2)$$

$$VU^2V = UV^2U, \quad (3)$$

$$UVU^2 = -U^2VU, \quad (4)$$

$$VU^2V^2U = 0. \quad (5)$$

The rank of the ring A_N is now is shown to be 18, and A_N has $\mathbb{D}$-basis

$$\{1, U, V, U^2, V^2, UV, VU,$$
$$U^2V, VU^2, UV^2, V^2U, V^2U^2, U^2V^2,$$
$$VUV, U^2VU, V^2UV,$$
$$U^2V^2U, V^2U^2V\}.$$

Also, *the augmentation ideal* $\omega(A)$ *is nilpotent of degree* 6.

- On how to prove $(x-1)^3 = 0$ *is satisfied by all elements of* G.

1. Express $C = [a,b] - 1$ in terms of the $\omega(A)$-basis as

$$\begin{aligned} C \;=\; & UV - VU - 6U^2V + UV^2 - 5VU^2 + 2V^2U \\ & U^2V^2 + 2V^2U^2 \\ & -4UVU + -U^2VU - VUV^2 \\ & +2U^2V^2U - VU^2V^2. \end{aligned}$$

2. Compute the table of monomials involving C in terms of the $\omega(A)$-basis, such as

$$UCU = 2U^2VU - U^2V^2U.$$

3. Express the basic commutators

$$[a,b,a]-1,[a,b,b]-1,$$
$$[a,b,a,a]-1,[a,b,b,a]-1,$$
$$[a,b,b,b]-1$$

in terms of the $\omega(A)$ basis.

4. Use the table of basic relations to rewrite $\left(a^i b^j - 1\right)^3$ in terms of the $\omega(A)$ basis, thus transforming the exponents into coefficients. The same is done successively for $g = a^i b^j [a,b]^k, a^i b^j [a,b]^k [a,b,a]^l [a,b,b]^p$ and commutators of weight 4 do not have any influence.

- To finish.

What about groups which satisfy $(x-1)^4 = 0$?

For characteristic 5 ?