Parabolic factorizations of Steinberg groups

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Contents

Principal notation

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Theorem (G. Taddei '83)

If Φ is of rank > 1 then $\mathrm{E}(\Phi, R) \trianglelefteq \mathrm{G}(\Phi, R)$.

Gauss factorization

For $H_1, \ldots, H_n \leq G$ denote by $H_1 \cdot \ldots \cdot H_n$ the subset of G consisting of all products of the form $h_1 \ldots h_n$, $h_i \in H_i$.

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For R of "stable rank 1" one has (Smolensky et al.)

$$\mathrm{E}(\Phi, R) = \mathrm{U}^-(\Phi, R) \cdot \mathrm{U}(\Phi, R) \cdot \mathrm{U}^-(\Phi, R) \cdot \mathrm{U}(\Phi, R).$$

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Factorizations of such type have no chance to be true for more general R (e.g. $\mathrm{E}(A_2, \mathbb{C}[x]) = \mathrm{E}(3, \mathbb{C}[x])$ does not have finite length with respect to $t_\alpha(\xi)$).

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Question

What weaker form of such decomposition survives for more general choice of R?

Stable rank

A collection $(a_1, \ldots, a_n)$ of elements of R is called *unimodular* if $a_1, \ldots, a_n$ span R as an ideal.

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Example

If R is semilocal $\mathrm{sr}(R) = 1$, $\mathrm{dimMax}(R) = 0$.
If R is euclidian (e.g. $\mathbb{Z}$ or $\mathbb{C}[x]$) $\mathrm{sr}(R) = 2$, $\mathrm{dimMax}(R) = 1$.

Contents

Dennis–Vaserstein factorization for $\mathrm{E}(n, R)$.

Assume that $\mathrm{sr}(R) \leq n - 2$. Then $g \in \mathrm{E}(n, R)$ can be decomposed as a product of the following form (with both $n-1 \times n-1$ blocks belonging to $\mathrm{E}(n-1, R)$).

$$g = \left(\begin{array}{c|ccc} 1 & * & \dots & * \\ \hline 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{array}\right) \cdot \left(\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \hline * & 0 & \dots & 1 \end{array}\right) \cdot \left(\begin{array}{ccc|c} * & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & * & * \\ \hline 0 & \dots & 0 & 1 \end{array}\right)$$

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Remark

We will formulate a little bit more precise result.

Steinberg groups, $\mathrm{K}_1(\Phi, R)$ and $\mathrm{K}_2(\Phi, R)$

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$$[t_\alpha(\xi_1), t_\beta(\xi_2)] = \prod_{i\alpha+j\beta\in\Phi,\ i,j>0} t_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi_1^i\xi_2^j), \quad \alpha \neq -\beta. \tag{2}$$

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Definition

Set $\pi(x_\alpha(\xi)) = t_\alpha(\xi)$ then $\mathrm{K}_1(\Phi, R)$, $\mathrm{K}_2(\Phi, R)$ are defined as

$$\mathrm{K}_2(\Phi, R) \hookrightarrow \mathrm{St}(\Phi, R) \xrightarrow[\pi]{} \mathrm{G}(\Phi, R) \twoheadrightarrow \mathrm{K}_1(\Phi, R)$$

Parabolic subgroups

- Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a system of simple roots of Φ (with standard enumeration).

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- $\mathrm{U}_i^- := \langle\{x_\alpha(\xi),\ m_i(\alpha) < 0\}\rangle,\ 1 \leq i \leq l.$

Reformulation of Dennis—Vaserstein theorem for $\mathrm{St}(A_\ell, R)$

With the notation introduced above Dennis–Vaserstein factorization theorem can be stated as follows

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Theorem (Dennis—Vaserstein)

For R such that $\mathrm{sr}(R) \leq \ell - 1$ one has

$$\mathrm{St}(A_\ell, R) = \mathrm{P}_1 \cdot (\mathrm{U}_1^- \cap \mathrm{U}_\ell^-) \cdot \mathrm{P}_\ell.$$

Application to stability

An embedding $\Psi \subseteq \Phi$ induces *stabilization maps*

$$\mathrm{K}_1(\Psi, R) \to \mathrm{K}_1(\Phi, R), \quad \mathrm{K}_2(\Psi, R) \to \mathrm{K}_2(\Phi, R)$$

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Corollary

For R such that $\mathrm{sr}(R) \leq \ell - 1$

$$\mathrm{K}_1(A_{\ell-1}, R) \to \mathrm{K}_1(A_\ell, R) \text{ is injective,}$$

$$\mathrm{K}_2(A_{\ell-1}, R) \to \mathrm{K}_2(A_\ell, R) \text{ is surjective.}$$

For analogues of Dennis—Vaserstein decompositions for other Chevalley groups and other stability theorems see
M. R. Stein, *Stability theorems for* K_1, K_2 *and related functors modeled on Chevalley groups.* Japan J. Math., **4**, 1 (1978), 77–108.

Contents

Stability theorem for embedding $E_6 \hookrightarrow E_7$

Theorem (E. Plotkin '98)

If $\mathrm{dimMax}(R) \leq 4$ *then* $\mathrm{K}_1(E_6, R) \to \mathrm{K}_1(E_7, R)$ *is surjective.*

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Question

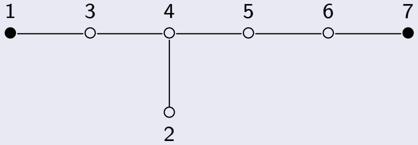
Can we describe the kernel of the stabilization map for $\mathrm{dimMax}(R) = 4$ (i.e. obtain a 'prestabilization' theorem)? Similar question for $A_{\ell-1} \hookrightarrow A_\ell$ has been studied by van der Kallen '87.

Parabolic factorization for E_ℓ, $\ell = 6, 7, 8$

Theorem

For R such that $\mathrm{asr}(R) \leq \ell - 2$ one has

$$\mathrm{St}(E_\ell, R) = \mathrm{P}_1 \cdot (\mathrm{U}_1^- \cap \mathrm{U}_\ell^-) \cdot \mathrm{P}_\ell, \;\; \ell = 6, 7, 8.$$



Prestabilization for $E_6 \hookrightarrow E_7$, $\mathrm{dimMax}(R) = 4$

$$\begin{tikzcd} D_5 \arrow[r] \arrow[d] & D_6 \arrow[d] \\ E_6 \arrow[r] & E_7 \end{tikzcd}$$

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\end{tikzcd}$$

Prestabilization for $E_6 \hookrightarrow E_7$, $\mathrm{dimMax}(R) = 4$

Every element of $\mathrm{G}(E_6, R)$ becoming elementary in $\mathrm{G}(E_7, R)$ modulo $\mathrm{E}(E_6, R)$ comes from an element of $\mathrm{G}(D_5, R)$ becoming elementary when embedded into $\mathrm{G}(D_6, R)$.

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Thank you for your attention!