SANDWICH CLASSIFICATION THEOREM

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1. Sandwich classification theorem

Let G be a group. For a subgroup D of a group G put

$$\mathrm{L}(D,G) = \{H \mid D \leqslant H \leqslant G\}.$$

Let $\mathcal{L}$ be a lattice of subgroups of G, satisfying some property. We say that $\mathcal{L}$ satisfies sandwich classification theorem if

$$\mathcal{L} = \bigsqcup \mathrm{L}(F_i, N_i) \qquad \text{and} \qquad F_i \triangleleft N_i,$$

where i ranges over some index set.

2. Subgroups containing a given subgroup

Let D be a subgroup of a group G. First, consider the lattice $\mathrm{L}(D,G)$.

For a subgroup $H \leqslant G$ denote by D^H the smallest subgroup, containing D and normalized by H. The normalizer of H in G is denoted by $N_G(H)$. A subgroup $H \in \mathcal{L}$ is called D-full if $D^H = H$.

Definition. We say that the lattice $\mathrm{L}(D,G)$ satisfies sandwich classification if for each subgroup $H \in \mathrm{L}(D,G)$ there exists a unique D-full subgroup F such that

$$F \leqslant H \leqslant N_G(F).$$

This is equivalent to saying that for any $H \leqslant G$ the subgroup D^H is D-full, i. e.

$$D^{D^H} = D^H.$$

Clearly, sandwich classification holds if D is normal in G or D is a maximal subgroup. More generally, if D is pronormal in G, then $\mathcal{L}$ satisfies sandwich classification.

3. Subgroups, normalized by a given subgroup

Now, let us consider the lattice $\mathcal{L}$ of subgroups of G, normalized by D. Denote $[D,H]$ the mutual commutator subgroup. A subgroup $H \in \mathcal{L}$ is called D-perfect if $[D,H] = H$. For $H \in \mathcal{L}$ denote by $C_{D,G}(H)$ the largest subgroup C of $N_G(H)$ satisfying $[C,D] \leqslant H$.

Definition. We say that the lattice $\mathcal{L}$ satisfies sandwich classification if for each subgroup $H \in \mathcal{L}$ there exists a unique D-perfect subgroup F such that

$$F \leqslant H \leqslant C_{D,G}(F).$$

This is equivalent to saying that for any $H \in \mathcal{L}$ the subgroup $[D,H]$ is D-perfect, i. e.

$$\big[[H,D],D\big] = [H,D].$$

Theorem 1. *Let D be a perfect subgroup (i.e. $[D,D] = D$) of a group G. Suppose that sandwich classification holds for subgroups, containing D. Then sandwich classification holds for subgroups, normalized by D.*

Denote by $\mathcal{P}_F$ the set of all F-perfect subgroups of F. Then the set of all D-perfect subgroups is a union of $\mathcal{P}_F$ over all D-full subgroups F of G.

4. Examples

Let R be a commutative ring and $n \geqslant 3$. For the following situations the lattice $\mathrm{L}(D, G)$ satisfies sandwich classification.

1. $G = \mathrm{GL}_n(R)$, D is the group of diagonal matrices, R is a field, containing at least 7 elements (Borewich, 1976).
   F_σ are net groups.
   $\mathcal{L}$ does not satisfy sandwich classification.
2. $G = \mathrm{GL}_n(R)$, $D = \mathrm{ESp}_n(R)$ or $D = \mathrm{EO}_n(R)$ (Vavilov, Petrov 2000–2007).
   $F_I = D \cdot \mathrm{E}_n(R, I)$, where I is an ideal of R.
   $\mathcal{L}$ satisfies sandwich classification but the normal structure of F_I's is unknown.
3. $G = \mathrm{GL}_n(R)$, D is an elementary block-diagonal group with dimensions of diagonal blocks $\geqslant 3$ (Borevich, Vavilov 1984).
   F_σ are net groups.
   Similar theorem for $\mathcal{L}$ is known but not published.

4.1. $G = \mathrm{GL}_n(R)$, $D = \mathrm{E}_n(K)$, where K is a Dedekind domain and R is its field of fractions (Shmidt 1979).

4.2. $G = \mathrm{GL}_n(R)$, $D = \mathrm{E}_n(K)$, where K is a field and R is its algebraic extension (Nuzhin 1983).

4.3. $G = \mathrm{Sp}_n(R)$, or $G = \mathrm{SO}_{2k+1}(R)$, $D = E(K)$ is the elementary subgroup over a subring $K \ni 1/2$. (Stepanov 2012).
$F_P = E(P)$, where P is a subring of R, containing K.
Sandwich classification for $\mathcal{L}$ follows from Theorem 1 and the normal structure of Chevalley groups (Abe, Taddei, Vaserstein 1986-1989). $F_{P,\mathfrak{q}} = E(P, \mathfrak{q})$, where $\mathfrak{q}$ is an ideal of P.

5. $D = \mathrm{E}_m(R) \otimes \mathrm{E}_k(R)$, where $mk = n$, $m - 2 \geqslant k \geqslant 3$ (Ananievski, Vavilov, Sinchuk 2009-2011-???).

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