Let $G$ be a group. For a subgroup $D$ of a group $G$ put

$$L(D, G) = \{ H \mid D \leq H \leq G \}.$$ 

Let $\mathcal{L}$ be a lattice of subgroups of $G$, satisfying some property. We say that $\mathcal{L}$ satisfies sandwich classification theorem if

$$\mathcal{L} = \bigsqcup L(F_i, N_i) \quad \text{and} \quad F_i \triangleleft N_i,$$

where $i$ ranges over some index set.
2. **Subgroups containing a given subgroup**

Let $D$ be a subgroup of a group $G$. First, consider the lattice $L(D, G)$. For a subgroup $H \leq G$ denote by $D^H$ the smallest subgroup, containing $D$ and normalized by $H$. The normalizer of $H$ in $G$ is denoted by $N_G(H)$. A subgroup $H \in \mathcal{L}$ is called $D$-full if $D^H = H$.

**Definition.** We say that the lattice $L(D, G)$ satisfies sandwich classification if for each subgroup $H \in L(D, G)$ there exists a unique $D$-full subgroup $F$ such that

$$F \leq H \leq N_G(F).$$

This is equivalent to saying that for any $H \leq G$ the subgroup $D^H$ is $D$-full, i.e.

$$D^{D^H} = D^H.$$

Clearly, sandwich classification holds if $D$ is normal in $G$ or $D$ is a maximal subgroup. More generally, if $D$ is pronormal in $G$, then $\mathcal{L}$ satisfies sandwich classification.
3. **Subgroups, normalized by a given subgroup**

Now, let us consider the lattice \( \mathcal{L} \) of subgroups of \( G \), normalized by \( D \). Denote \([D, H]\) the mutual commutator subgroup. A subgroup \( H \in \mathcal{L} \) is called \( D \)-perfect if \([D, H] = H\). For \( H \in \mathcal{L} \) denote by \( C_{D,G}(H) \) the largest subgroup \( C \) of \( N_G(H) \) satisfying \([C, D] \leq H\).

**Definition.** We say that the lattice \( \mathcal{L} \) satisfies sandwich classification if for each subgroup \( H \in \mathcal{L} \) there exists a unique \( D \)-perfect subgroup \( F \) such that

\[
F \leq H \leq C_{D,G}(F).
\]

This is equivalent to saying that for any \( H \in \mathcal{L} \) the subgroup \([D, H]\) is \( D \)-perfect, i.e.

\[
[[H, D], D] = [H, D].
\]

**Theorem 1.** Let \( D \) be a perfect subgroup (i.e. \([D, D] = D\)) of a group \( G \). Suppose that sandwich classification holds for subgroups, containing \( D \). Then sandwich classification holds for subgroups, normalized by \( D \).

Denote by \( \mathcal{P}_F \) the set of all \( F \)-perfect subgroups of \( F \). Then the set of all \( D \)-perfect subgroups is a union of \( \mathcal{P}_F \) over all \( D \)-full subgroups \( F \) of \( G \).
4. Examples

Let \( R \) be a commutative ring and \( n \geq 3 \). For the following situations the lattice \( L(D, G) \) satisfies sandwich classification.

1. \( G = \text{GL}_n(R) \), \( D \) is the group of diagonal matrices, \( R \) is a field, containing at least 7 elements (Borevich, 1976).
   \( F_\sigma \) are net groups.
   \( L \) does not satisfy sandwich classification.

2. \( G = \text{GL}_n(R) \), \( D = \text{ESp}_n(R) \) or \( D = \text{EO}_n(R) \) (Vavilov, Petrov 2000–2007).
   \( F_I = D \cdot E_n(R, I) \), where \( I \) is an ideal of \( R \).
   \( L \) satisfies sandwich classification but the normal structure of \( F_I \)’s is unknown.

3. \( G = \text{GL}_n(R) \), \( D \) is an elementary block-diagonal group with dimensions of diagonal blocks \( \geq 3 \) (Borevich, Vavilov 1984).
   \( F_\sigma \) are net groups.
   Similar theorem for \( L \) is known but not published.

4.1. \( G = \text{GL}_n(R) \), \( D = E_n(K) \), where \( K \) is a Dedekind domain and \( R \) is its field of fractions (Shmidt 1979).

4.2. \( G = \text{GL}_n(R) \), \( D = E_n(K) \), where \( K \) is a field and \( R \) is its algebraic extension (Nuzhin 1983).

4.3. \( G = \text{Sp}_n(R) \), or \( G = \text{SO}_{2k+1}(R) \), \( D = E(K) \) is the elementary subgroup over a subring \( K \ni 1/2 \). (Stepanov 2012).
   \( F_P = E(P) \), where \( P \) is a subring of \( R \), containing \( K \).
   Sandwich classification for \( L \) follows from Theorem 1 and the normal structure of Chevalley groups (Abe, Taddei, Vaserstein 1986–1989). \( F_{P, q} = E(P, q) \), where \( q \) is an ideal of \( P \).
5. \( D = E_m(R) \otimes E_k(R) \), where \( mk = n, m - 2 \geq k \geq 3 \) (Ananievski, Vavilov, Sinchuk 2009-2011-???).

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