GENERALIZING THE CONCEPT OF QUASINORMALITY

In 1937, Itô introduced the concept of *quasinormal* subgroups. A subgroup H of a group G is *quasinormal* in G if $HK = KH$ for all subgroups K of G, i.e. $\langle H, K\rangle = HK$. Write H *qn* G. Sometimes they have been called *permutable* subgroups. One of their most important properties is the following, due to Ore:-

In finite groups, quasinormal subgroups are always subnormal.

In fact they are always *ascendant* in infinite groups, indeed in at most $\omega + 1$ steps. Napolitani & I proved this independently many years ago & neither of us published the result, probably because all known examples were ascendant in at most ω steps. Also simple groups have no proper non-trivial quasinormal subgroups.

In 1962, Itô & Szèp proved that

G finite & H qn G implies that H/H_G is nilpotent.

Rather curiously, even as recently as 1967, the only core-free examples that had appeared were abelian. Then in 1967, John Thompson found examples in finite p-groups (p odd) of class 2. Six years later, Maier & Schmid proved the following:-

G finite & H qn G implies that H/H_G lies in the hypercentre of G/H_G,

improving the Itô-Szèp result considerably.

The climax appeared in 1982, due to Berger & Gross:-

For each prime p & integer n, there is a finite p-group $G = HX$ with H qn G, $H_G = 1$, exponent of $H = p^{n-1}$ & X cyclic of order p^n. And these groups are universal in the sense that any finite p-group, with a similar factorization into subgroups with the same properties, embeds in G.

But then in 2010, John Cossey & I showed that these groups *G* have remarkably few quasinormal subgroups lying in *H.* When *p* is odd, they can have exponent only p, p^{n-2} & p^{n-1}. This is rather unfortunate, because quasinormal subgroups of finite *p*-groups are invariant under projectivities, i.e. subgroup lattice isomorphisms. (Normal subgroups are not.) So quasinormal subgroups don't tell us much about the subgroup lattices in these Berger-Gross groups. Why are *qn*-subgroups of finite *p*-groups invariant under projectivities? The point is they are always modular, i.e. satisfy the modular identity. Indeed *a subgroup of a finite group is quasinormal if & only if it is modular & subnormal*. Thus in a finite *p*-group, the concepts of being quasinormal & modular are the same. And modular subgroups are obviously invariant under projectivities. So *qn*-subgroups are surely relevant when the structure of a finite *p*-group is approached via its subgroup lattice. Therefore we need to generalise the concept of quasinormality.

We assume that G is a finite p-group.

The reason for this is, when *H qn G*, the complexities of the embedding of *H* in *G* reduce to the case where *G* is a *p*-group, i.e. when *H* is a *modular* subgroup. This follows from the Maier-Schmid result. For, *G* finite & *H qn G* with $H_G = 1$ implies that each Sylow *p*-subgroup of *P* of *H* is quasinormal in *G* & each *p'*-element of *G* commutes with each element of *P.*

So let *G* be a finite *p*-group & *H* a subgroup of *G* s.t. for all subgroups *K* of *G*

$$\langle H, K\rangle = HKH.$$

Then in fact *H qn G.* For, there is a central element of $\langle H, K\rangle$ of the form *hk* and h & k commute. Then, by induction, $G = HK\langle hk\rangle = H\langle hk\rangle K = HK$. Therefore we say

H is <u>4-quasinormal</u> in G if $\langle H, K\rangle = HKHK$ for all cyclic subgroups K.

Write $H\ qn_4\ G$. When defining quasinormality, cyclic K is sufficient. If we allow *all* subgroups K here, then we will say that H is *<u>strongly 4-quasinormal</u>*. What can we say about 4-quasinormal subgroups? Are they invariant under projectivities? There is one very simple observation to make at the start.

Every subgroup of a nilpotent group of class at most 2 is 4-qn.

For, $\langle H, K\rangle = H[H, K]K = HKHK$ if K is cyclic.

In the case of a *qn*-subgroup H, a lot of information has been discovered *when H is abelian*, & even more *when H is cyclic*. Indeed in the latter case, all the subgroups of H are also quasinormal in G. If also G is a finite p-group (p odd), then Cossey & I proved that

$[H, G]$ *is abelian & H acts on it as a group of power automorphisms.*

When H is an abelian quasinormal subgroup of G, then again much can be said. For any G (finite or infinite), it follows that H^n *qn G if n is odd or if* 4 *divides n* (Cossey, Stonehewer & Zacher). Also when $G = HX$ (a finite p-group) with H abelian & X cyclic, then *there are* 2 *canonical composition series of G passing through H with all the terms quasinormal in G.* One is a refinement of the ascending Ω-series, the other is a refinement of the descending ℧-series. Thus we start by considering cyclic 4-*qn* subgroups.

Suppose that $H\ qn_4\ G = \langle H, K\rangle$, a finite p-group ($p \geq 5$), with H & K cyclic. When H & K have order p, then $|G| \leq p^3$. So

G is either elementary abelian of rank ≤ 2 *or extraspecial of order* p^3. (1)

Let $|H| = \langle h \rangle$ of order p^m, $K = \langle k \rangle$ of order p^n, $C = \langle [h, k] \rangle$ of order p^r. Then by (1), $|G/G^p| \leq p^3$. Since $3 < p - 1$, G is regular (see Huppert vol.1). Then it follows easily that Ω_1 of H, K or C is normal in G & induction on $|G|$ shows that

$$G = HCK. \qquad (2)$$

Also *any p-group with this structure has all its subgroups* 4-*qn*. Thus, as in the case of cyclic quasinormal subgroups, we have:-

<u>THEOREM</u> 1. *Every subgroup of a cyclic* qn_4*-subgroup of a finite p-group G is* 4-*qn in G*.

It seems that there are not many groups like G given by (2). For, there is a unique normal subgroup N of G maximal s.t. $N \subset HK$. Let $N = 1$, i.e. factor G by N. Then we must have $H_G = K_G = 1$ and it follows easily that $m = n = r$. Also G has rank 3. Just infinite pro-p-groups are relevant here, i.e. inverse limits of finite p-groups that are infinite & have all non-trivial closed normal subgroups of finite index. Suppose that $\gamma_3(G) = G^p$. Then it turns out that the lower central factors of G are elementary abelian of ranks 2 & 1 alternating. Also G has width 2 & obliquity 0 (i.e. the normal subgroups occur only between adjacent terms of the lower central series). In fact there are precisely 2 just infinite pro-p-groups of rank 3, width 2 & obliquity 0. They come from the 2 central simple algebras of dimension 4 over the quotient field of the p-adic integers (i.e. the field is the centre). Call these groups G_1 & G_2. Then our group G is isomorphic to $G_i/\gamma_{2m+1}(G_i)$, $i = 1$ or 2. (See The Structure of Groups of Prime Power Order, Leedham-Green & McKay.) As m increases, the derived length here increases. However, the above structure derived from (2) allows us to establish a positive result concerning projectivities:-

THEOREM 2. *Cyclic qn_4-subgps of finite p-gps are invariant under projectivities.*

To see this, one reduces to the case $m = n = r$. Then $G^pC = X$, say, is normal in G & by regularity, the p^{m-1}th power of X is $\Omega_1(C)$ which is normal in G. The same is true in the projective image of G. So induction on order applies.

Finally we have

THEOREM 3. *Cyclic qn_4-subgps of finite p-gps are strongly* 4-*qn*.

That is $<H, K> = HKHK$ for *all* subgps K. The argument is similar to that of Theorem 2.

The early part of this work was done in collaboration with John Cossey in Canberra & Warwick.

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April 2014