Classical Hurwitz groups of low rank

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Dedicated to David Chillag

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Several authors contributed to this investigation, like Conder, Zalesski, Malle, R. Wilson, Woldar, Vsemirnov, T., Lucchini, J. Wilson, and many others.

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The low-rank case is more difficult and requires different methods, based on the knowledge of characters and maximal subgroups.

My talk will concentrate concentrate on this case.

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By Scott's formula, applied to the conjugation action of $\mathrm{GL}_n(\mathbb{F})$ on $M := \mathrm{Mat}_n(\mathbb{F})$:

$$\dim\left(C_M(x)\right) + \dim\left(C_M(y)\right) + \dim\left(C_M(xy)\right) \;\leq\; n^2 + 2. \qquad (1)$$

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More explicitly, if the triple (x, y, xy) is rigid and $x', y', x'y'$ are respectively conjugate to x, y, xy, there exists $g \in \mathrm{GL}_n(\mathbb{F})$ s.t.:

$$x' = x^g, \quad y' = y^g.$$

In particular $\langle x', y' \rangle$ is conjugate to $\langle x, y \rangle$.

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List of Hurwitz simple groups of dimension $n \leq 5$.

- $\mathrm{PSL}_2(p^{m_2})$ (Macbeath);
- $\mathrm{PSL}_5(p^{m_5})$, if m_5 is odd and $p \neq 5$; (Zalesski, T.)
- $\mathrm{PSU}_5(p^{m_5})$, if m_5 is even and $p \neq 5$.

$$\begin{cases} m_2 := \mathrm{order}(p^2) \pmod 7 \text{ if } p \neq 7 \\ m_2 := 1 \text{ if } p = 7. \end{cases}$$

$$\begin{cases} m_5 := \mathrm{order}(p) \pmod 5 \times \mathrm{order}(p^2) \pmod 7 \text{ if } p \neq 5, 7 \\ m_5 := 4 \text{ if } p = 7. \end{cases}$$

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$$(x, y, xy) \in \mathrm{SL}_n(\mathbb{F})^3, \ \ n \leq 7 \tag{2}$$

such that x^2, y^3 and $(xy)^7$ are scalar matrices and the group $\langle x, y\rangle$ is irreducible;

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(A different approach to this classification is also sketched in a paper of Plesken and Robertson, but without the elimination of singularities and reducible triples).

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- $t^6 + t^5 + t^4 + t^3 + t^2 + t + 1$ for xy.

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Every $g \in \mathrm{GL}_n(\mathbb{F})$ acts on the space Λ^2 of skew-symmetric matrices $A \in \mathrm{Mat}_n(\mathbb{F})$ with zero diagonals via:

$$A \mapsto gAg^T$$

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From family (IIa) we obtain the 1-parametric family:

$$x = \begin{pmatrix} 0 & 0 & -1 & 0 & r & -1 \\ 0 & 0 & 0 & -1 & 0 & a \\ 1 & 0 & 0 & 0 & -1 & -r \\ 0 & 1 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & c \\ 0 & 1 & 0 & 0 & b & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \tag{3}$$

$$a = \frac{2r^2-7r+7}{r^2-3r+3}, \quad b = \frac{-r^3+7r^2-16r+13}{r^2-3r+3}, \quad c = \frac{-r^3+2r^2-r-1}{r^2-3r+3}.$$

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The main of these conditions are:

1) $r^2 - 3r + 3 \neq 0$ (for the definition of y);

2) r is not a root of a certain polynomial of degree 12 (for the irreducibilty of $\langle x, y \rangle$);

3) $\mathbb{F}_{p^m} = \mathbb{F}_p \left[r^2 - 3r + 3 \right]$ (for the minimal field of definition).

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Theorem

Let $q \geq 5$, q odd. There exists $r \in \mathbb{F}_q$ satisfying our list of conditions. In particular $\mathrm{PSp}_6(q)$ is a Hurwitz group.

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- $\mathrm{PSp}_6(2^m)$ and $\mathrm{PSp}_6(3)$ are not Hurwitz (Vincent and Zalesski);
- for p odd, $\mathrm{Sp}_6(p^m)$ is not even $(2, 3)$-generated (Vsemirnov).

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$$\mathrm{G}_2(q),\ \ q \neq 2, 3, 4, \quad {}^2\,\mathrm{G}_2(q),\ \ q \neq 3^{2m+1}$$

We have evidence that, for $q \geq 8$, an appropriate choice of $r \in \mathbb{F}_q$, the matrices in (3) generate $G_2(q)$.

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They have been completely analyzed, obtaining only the following Hurwitz simple groups:

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- $\mathrm{PSL}_6(p^{m_6})$ if m_6 is odd and $p \neq 3$;
- $\mathrm{PSU}_6(p^{m_6})$ if m_6 is even and $p \neq 3$;

$m_6 := \mathrm{order}(p) \pmod 9$.

Rank 7.

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Application of Scott's formula to the space of symmetric matrices (viewed as a $GL_7(\mathbb{F})$-module via the action described above) and its dual gives that $\langle x, y \rangle$, when irreducible, preserves a non-degenerate symmetric matrix.

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So, analising these triples, we can only obtain subgroups of orthogonal groups.

All the rigid triples have been analyzed, giving only the following Hurwitz simple groups:
$m_7 := \mathrm{order}(p) \pmod{49}$.

- $\mathrm{PSL}_7(p^{m_7})$ if m_7 is odd and $p \neq 7$;

Classical simple groups which are not $(2,3)$-generated

- $\mathrm{PSL}_2(9)$
- $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(9)$, $\mathrm{PSU}_3(25)$,
- $\mathrm{PSL}_4(2)$, $\mathrm{PSU}_4(4) \simeq \mathrm{PSp}_4(3)$, $\mathrm{PSU}_4(9)$, $\mathrm{PSp}_4(p^k)$, $p = 2, 3$
- $\mathrm{PSU}_5(4)$
- $\Omega_8^+(2)$, $\mathrm{P}\Omega_8^+(3)$

This list is

certainly complete for classical groups of degree $n \leq 5$ (joint work with Pellegrini and Vsemirnov);

almost complete for $\mathrm{PSL}_n(q)$ (work of T. , joint work of Di Martino and Vavilov, more recent work of Tabakov and Tchakerian).