On Stallings decomposition theorem for pro-p groups (joint work with P. Zalesskii)

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Cayley graphs

and rays ...

- Let G be a discrete group being generated by a finite, symmetric generating set $S \subset G$ not containing 1, i.e., $S = S^{-1}$ and $1 \notin S$.
- Then $\Gamma = (\mathcal{V}, \mathcal{E})$ given by

$$\begin{aligned} \mathcal{V} &= G, \\ \mathcal{E} &= \{\, (g, gs) \mid g \in G,\ \ s \in S \,\}, \end{aligned}$$

  is called the **Cayley graph** associated with (G, S).
- Let $\mathfrak{p} = (\mathbf{e}_k)_{k \geq 0}$ be an **infinite path** in Γ without backtracking, i.e., $t(\mathbf{e}_k) = o(\mathbf{e}_{k+1})$ and $\bar{\mathbf{e}}_{k+1} \neq \mathbf{e}_k$ for all $k \geq 0$.
- For $m \geq 0$ define $\mathfrak{p}[m] = (\mathbf{e}_{k+m})_{k \geq 0}$.
- Put $\mathfrak{p} \sim \mathfrak{q}$ if there exist $m, n \geq 0$ such that $\mathfrak{p}[m] = \mathfrak{q}[n]$.
- An equivalence class $[\mathfrak{p}]$ (with respect to $\sim$) of infinite paths without backtracking is called a **ray**.

Cayley graphs

and ends . . .

- For a finite symmetric set of edges $\mathcal{R} \subseteq \mathcal{E}$ define $\Gamma_{\mathcal{R}} = (\mathcal{V}, \mathcal{E} \setminus \mathcal{R})$ ($\mathcal{R}$ symmetric $\iff \bar{\mathcal{R}} = \mathcal{R}$).
- A ray $[\mathfrak{p}]$ is said **to go to** ∞ if for all finite symmetric set of edges $\mathcal{R} \subseteq \mathcal{E}$ there exists $m = m(\mathcal{R}) \geq 0$ such that $\mathfrak{p}[m]$ is a path in $\Gamma_{\mathcal{R}}$.
- Let $[\mathfrak{p}]$, $[\mathfrak{q}]$ be rays going to ∞. Define $[\mathfrak{p}] \approx [\mathfrak{q}]$, if for all $\mathcal{R} \subseteq \mathcal{E}$ finite and symmetric and for all $m, n \geq 0$ such that $\mathfrak{p}[m]$ and $\mathfrak{p}[n]$ are infinite paths in $\Gamma_{\mathcal{R}}$, $\mathfrak{p}[m]$ and $\mathfrak{q}[n]$ are running in the same connectedness component of $\Gamma_{\mathcal{R}}$.
- The set of equivalence classes (with respect to $\approx$) of rays going to ∞ is called the **space of ends** $\mathrm{Ends}(\Gamma)$ of Γ.

The number of ends of a finitely generated group

Proposition

Let G be a finitely generated, infinite group with finite, symmetric generating system $S \subset G$. Then

$$\begin{aligned}\mathrm{card}(\mathrm{Ends}(\Gamma)) &= 1 + \mathrm{rk}_{\mathbb{Z}}(H^1(G, \mathbb{Z}[G])) = 1 + \dim_{\mathbb{F}_p}(H^1(G, \mathbb{F}_p[G])) \\ &= \dim_{\mathbb{R}}(H^1_c(|\Gamma|, \mathbb{R}))\end{aligned}$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, p prime, and $H^1_c(\_, \mathbb{R})$ denotes cohomology with compact support.

Definition

For a finitely generated group G the number

$$e(G) = \mathrm{card}(\mathrm{Ends}(\Gamma)) \in \mathbb{N}_0 \cup \{\infty\}$$

where $\Gamma = \Gamma(G, S)$ is a Cayley graph for a finite symmetric generating system S not containing 1, is called the **number of ends** of G.

Stallings' decomposition theorem . . .

- $e(G) \in \{0, 1, 2, \infty\}$;
- $e(G) = 0 \Leftrightarrow G$ finite;
- $e(G) = 2 \Leftrightarrow G$ virtually cyclic.

Theorem (J.R. Stallings (1971))

Let G be a finitely generated group satisfying $e(G) = \infty$. Then **either**

- $G \simeq A \coprod_C B$, *for some* $A, B, C \subseteq G$, $A, B \neq \{1\}$, C *finite;* **or**
- $G \simeq \mathrm{HNN}_B(A, t)$, *for some* $B \subsetneq A \subseteq G$, B *finite.*

Theorem (J.R. Stallings (1968), R.W. Swan (1969))

Let G be a (discrete) group satisfying $\mathrm{cd}_{\mathbb{Z}}(G) \leq 1$. Then G is a free group.

Theorem (A. Karrass - A. Pietrowski - D. Solitar (1973))

Let G be a finitely generated virtually free group. Then $G \simeq \pi_1(\mathcal{A}, \Lambda, x_0)$ for some finite graph of finite groups $\mathcal{A}$ based on a finite graph Λ.

The number of $\mathbb{F}_p$-ends of a pro-p group

ala O.V. Mel'nikov and A.A. Korenev ...

- Let G be a profinite group, and let

$$\mathbb{F}_p[\![G]\!] = \varprojlim_U \mathbb{F}_p[G/U],$$
$$\mathbb{Z}_p[\![G]\!] = \varprojlim_U \mathbb{Z}_p[G/U],$$

where the inverse limit is running over all open normal subgroups of G, denote the **completed** $\mathbb{F}_p$**- and** $\mathbb{Z}_p$**-algebra** of G, respectively.

Definition (O.V. Mel'nikov)

The **number of** $\mathbb{F}_p$**-ends** $\mathbf{E}(G)$ of a pro-p group G is defined by

$$\mathbf{E}(G) = 1 - \dim_{\mathbb{F}_p} H^0(G, \mathbb{F}_p[\![G]\!]) + \dim_{\mathbb{F}_p}(H^1(G, \mathbb{F}_p[\![G]\!])),$$

where $H^\bullet(G, \_)$ denotes **continuous cochain cohomology** (à la J. Tate).

Korenev's theorem

- $$\dim_{\mathbb{F}_p}(H^0(G,\mathbb{F}_p[\![G]\!])) = \mathrm{rk}_{\mathbb{Z}_p}(H^0(G,\mathbb{Z}_p[\![G]\!])) = \begin{cases} 1 & \text{for } |G| < \infty, \\ 0 & \text{for } |G| = \infty. \end{cases}$$
- In particular, $\mathbf{E}(G) = 0$ if, and only if, $|G| < \infty$.

Theorem (A.A. Korenev (2004))

Let G be a finitely generated pro-p group. Then

- $\mathbf{E}(G) \in \{0, 1, 2, \infty\}$*;*
- $\mathbf{E}(G) = 2$ *if, and only if, G is infinite virtually cyclic.*

The number of $\mathbb{Z}_p$-ends of a pro-p group

a slightly different approach ...

Definition

The **number of $\mathbb{Z}_p$-ends $\mathbf{e}(G)$** of a pro-p group G is defined by

$$\mathbf{e}(G) = 1 - \mathrm{rk}_{\mathbb{Z}_p} H^0(G, \mathbb{Z}_p[\![G]\!]) + \mathrm{rk}_{\mathbb{Z}_p}(H^1(G, \mathbb{Z}_p[\![G]\!])).$$

Theorem (T.W. & P. Zalesskii (2013))

Let G be a finitely generated pro-p group. Then

- $\mathbf{e}(G) \in \{0, 1, 2, \infty\}$.
- $\mathbf{e}(G) = 0$ *if, and only if, G is a finite p-group.*
- $\mathbf{e}(G) = 2$ *if, and only if, G is an infinite virtually cyclic pro-p group.*
- $\mathbf{e}(G) \leq \mathbf{E}(G)$.

- From a preprint of K. Wingberg's follows that $\mathbf{e}(G) = \mathbf{E}(G)$.

Stallings' decomposition theorem

for finitely generated pro-p groups . . .

Theorem (T.W. & P. Zalesskii (2013))

Let G be a finitely generated pro-p group satisfying $\mathbf{e}(G) = \infty$. Then **either**

- $G \simeq A \coprod_C B$*, for some* $A, B, C \subseteq G$*,* $A \neq C \neq B$*,* C *finite;* **or**
- $G \simeq \mathrm{HNN}_B(A, t)$*, for some* $B \subsetneq A \subseteq G$*,* B *finite.*

Hilbert's "Principal Ideal Conjecture"

- Let L/K be a finite extension of number fields.
- Then $\mathcal{O}_K$ and $\mathcal{O}_L$ are Dedekind domains,
- and $\mathrm{rk}_{\mathcal{O}_K}(\mathcal{O}_L) = |L : K|$.
- If $\mathfrak{a} \triangleleft \mathcal{O}_K$, then $\mathcal{O}_L\mathfrak{a} \triangleleft \mathcal{O}_L$.

Conjecture (D. Hilbert (1892))

Let K be a number field, and let $H(K)$ be its Hilbert class field. Then for any $\mathfrak{a} \triangleleft \mathcal{O}_K$, $\mathcal{O}_{H(K)}\mathfrak{a}$ is a principal ideal in $\mathcal{O}_{H(K)}$.

and Ph. Furtwängler's solution:

the transfer vanishing theorem ...

Theorem (Ph. Furtwängler (1929))

Hilbert's Principal Ideal Conjecture is true.

- Let G be a finite group, and $G^{\mathrm{ab}} = G/[G,G]$.
- Let H be a subgroup, and let $\mathcal{R} \subseteq G$ be a set of representatives of G/H.
- Then $Tr_{G,H}\colon G^{\mathrm{ab}} \to H^{\mathrm{ab}}$, $Tr_{G,H}(g[G,G]) = \prod_{r\in R} rgr^{-1}[H,H]$ is a $\mathbb{Z}$-linear map - the **transfer from G to H.**

Theorem (Ph. Furtwängler (1929))

Let G be a finite metabelian group. Then

$$Tr_{G,[G,G]}\colon G^{\mathrm{ab}} \longrightarrow [G,G]^{\mathrm{ab}}$$

is the 0-map.

End groups

introducing a kind of "geometric" concept ...

- For a pro-p group G, let $\Phi(G)$ denote the **Frattini group** of G, i.e.,

$$G^{\mathrm{ab,el}} = G/\Phi(G)$$

  is the **maximal elementary abelian quotient** of G.
- Let $G^{\mathrm{ab}} = G/\mathrm{cl}([G,G])$ denote the **maximal abelian quotient**, and put

$$G^{\mathrm{tf,el}} = G^{\mathrm{ab}}/(pG^{\mathrm{ab}} + \mathrm{tor}(G^{\mathrm{ab}})),$$

  where $\mathrm{tor}(G)$ denotes the closure of all torsion elements of the compact abelian group G^{ab}.
- Define the $\mathbb{F}_p$**-end group** and $\mathbb{Z}_p$**-end group** of G by

$$\partial\mathbf{E}(G) = \varinjlim_U U^{\mathrm{ab,el}}, \qquad \text{and} \qquad \partial\mathbf{e}(G) = \varinjlim_U U^{\mathrm{tf,el}},$$

  where the maps in the direct limits are given by the transfer.

Properties of the end groups

and some canonical maps

- By construction, one has canonical maps and a commutative diagram

$$\begin{tikzcd} & G^{\mathrm{ab,el}} \arrow[dl, "j_G"'] \arrow[dr, "j_G^{\mathrm{tf}}"] & \\ \partial\mathbf{E}(G) \arrow[rr, "\pi_G"] & & \partial\mathbf{e}(G) \end{tikzcd}$$

  where π_G is surjective.
- If G is finitely generated, one has canonical isomorphisms

$$\partial\mathbf{E}(G) \simeq H^1(G, \mathbb{F}_p[\![G]\!])^\vee,$$
$$\partial\mathbf{e}(G) \simeq \mathrm{im}(H^1(G, \mathbb{Z}_p[\![G]\!]) \to H^1(G, \mathbb{F}_p[\![G]\!]))^\vee,$$

  where $\_^\vee$ denotes the **Pontryagin dual**.
- If G is infinite, then $\mathbf{E}(G) = 1 + \dim_{\mathbb{F}_p}(\partial\mathbf{E}(G))$ and $\mathbf{e}(G) = 1 + \dim_{\mathbb{F}_p}(\partial\mathbf{e}(G))$.

Semi-direct factors isomorphic to $\mathbb{Z}_p$

and the end groups . . .

- Let G be a pro-p group.
- A sequence of morphisms of pro-p groups $\delta\colon G \xrightarrow{\pi} \mathbb{Z}_p \xrightarrow{\sigma} G$ is called a **semi-direct factor isomorphic to $\mathbb{Z}_p$** if
  - π is surjective, and
  - $\pi \circ \sigma = \mathrm{id}_{\mathbb{Z}_p}$.
- The semi-direct factor δ isomorphic to $\mathbb{Z}_p$ is called an **$\mathbb{F}_p$-direction**, if $j_G(\sigma(1)\Phi(G)) \neq 0$, and a **$\mathbb{Z}_p$-direction**, if $j_G^{\mathrm{tf}}(\sigma(1)\Phi(G)) \neq 0$.

Pro-p groups with an $\mathbb{F}_p$-direction

Theorem (T.W. (2013))

Let G be a pro-p group, and let $\delta\colon G \xrightarrow{\pi} \mathbb{Z}_p \xrightarrow{\sigma} G$ be an $\mathbb{F}_p$-direction. Put $\Sigma = \mathrm{im}(\sigma)$, $N = \mathrm{cl}(\langle {}^g\Sigma \mid g \in G \rangle)$, and let $\bar{G} = G/N$. Then

- *N is a free pro-p group;*
- *$N/\Phi(N) \simeq \mathbb{F}_p[\![\bar{G}]\!]$;*
- *if G is countably based, then*

  $$\{1\} \longrightarrow N/\Phi(N) \longrightarrow G/\Phi(N) \longrightarrow \bar{G} \longrightarrow \{1\}$$

  is a split extension of pro-p groups.
- *If the extension*

  $$\{1\} \longrightarrow N \longrightarrow G \longrightarrow \bar{G} \longrightarrow \{1\}$$

  splits, then $G \simeq \mathbb{Z}_p \amalg \bar{G}$.

Finitely generated pro-p groups with a $\mathbb{Z}_p$-direction

Theorem (T.W. & P. Zalesski (2013))

Let G be a finitely generated pro-p group, and let $\delta\colon G \xrightarrow{\pi} \mathbb{Z}_p \xrightarrow{\sigma} G$ be a $\mathbb{Z}_p$-direction. Put $\Sigma = \mathrm{im}(\sigma)$, $N = \mathrm{cl}(\langle\, {}^g\Sigma \mid g \in G\,\rangle)$, and let $\bar{G} = G/N$. Then

- *the extension of pro-p groups*

$$\{1\} \longrightarrow N \longrightarrow G \longrightarrow \bar{G} \longrightarrow \{1\}$$

  splits, i.e.,

- $G \simeq \mathbb{Z}_p \amalg \bar{G}$.

Complementary modules ...

- Let G be a finitely generated pro-p group, and
- let $\delta\colon G \xrightarrow{\pi} \mathbb{Z}_p \xrightarrow{\sigma} G$ be a $\mathbb{Z}_p$-direction.
- Put $\Sigma = \mathrm{im}(\sigma)$, $s = \sigma(1)$ and let $N = \mathrm{cl}(\langle {}^{g}\Sigma \mid g \in G \rangle)$.
- Then one has canonical maps $\mathbb{Z}_p[\![G]\!] \xrightarrow{\alpha} \mathbb{Z}_p[\![G/\Sigma]\!] \xrightarrow{\beta} \mathbb{Z}_p$.
- The left $\mathbb{Z}_p[\![G]\!]$-module M is called a **complementary module** of δ, if there exist maps

$$\underline{\eta}\colon M \to \mathbb{Z}_p, \qquad \underline{\xi}\colon \mathbb{Z}_p[\![G]\!] \to M, \qquad \underline{j}\colon \ker(\beta) \to M,$$

such that the subsequent diagram is exact with exact rows ...

. . . and complementary short exact sequences

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathbb{Z}_p[\![G]\!] \arrow[r, equal] \arrow[d, "s-1"] & \mathbb{Z}_p[\![G]\!] \arrow[d, dotted, "\omega"] & \\
0 \arrow[r] & \ker(\beta) \arrow[r, dashed, "j"] \arrow[d, equal] & \mathbb{Z}_p[\![G]\!] \arrow[r, dashed, "\xi"] \arrow[d, "\alpha"] & M \arrow[r] \arrow[d, dashed, "\eta"] & 0 \\
0 \arrow[r] & \ker(\beta) \arrow[r, "\iota"] & \mathbb{Z}_p[\![G/\Sigma]\!] \arrow[r, "\beta"] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}
$$

- Moreover, $0 \longrightarrow \mathbb{Z}_p[\![G]\!] \xrightarrow{\omega} M \xrightarrow{\eta} \mathbb{Z}_p \longrightarrow 0$ will be called a **complementary short exact sequence**.

Existence

Lemma

Let $\delta\colon G \xrightarrow{\pi} \mathbb{Z}_p \xrightarrow{\sigma} G$ *be a* $\mathbb{Z}_p$*-direction of* G*. Then the canonical map* $\beta_*\colon \mathrm{Ext}^1_G(\mathbb{Z}_p, \mathbb{Z}_p[\![G]\!]) \to \mathrm{Ext}^1_G(\mathbb{Z}_p[\![G/\Sigma]\!], \mathbb{Z}_p[\![G]\!])$ *is surjective.*

$$\begin{tikzcd}
 & & \mathrm{Hom}_G(\ker(\beta), \mathbb{Z}_p[\![G]\!]) \arrow[d, "\alpha\circ."] \\
 & \mathrm{End}_G(\mathbb{Z}_p[\![G/\Sigma]\!]) \arrow[r, "\chi"] \arrow[d] & \mathrm{Hom}_G(\ker(\beta), \mathbb{Z}_p[\![G/\Sigma]\!]) \arrow[d, "\gamma"] \\
\mathrm{Ext}^1_G(\mathbb{Z}_p, \mathbb{Z}_p[\![G]\!]) \arrow[r, "\beta_*"] & \mathrm{Ext}^1_G(\mathbb{Z}_p[\![G/\Sigma]\!], \mathbb{Z}_p[\![G]\!]) \arrow[r, "0"] & \mathrm{Ext}^1_G(\ker(\beta), \mathbb{Z}_p[\![G]\!])
\end{tikzcd}$$

In particular, $\chi(\mathrm{id}_{\mathbb{Z}_p[\![G/\Sigma]\!]}) = \iota$, and, by the lemma, $\gamma(\iota) = 0$. Hence there exists $\underline{j} \in \mathrm{Hom}_G(\ker(\beta), \mathbb{Z}_p[\![G]\!])$ such that $\iota = \alpha \circ \underline{j}$. As ι is injective, $\underline{j}$ is injective.

Lattices and permutation modules

Definition

Let G be a finite group. A left $\mathbb{Z}_p[G]$-module M is called a left $\mathbb{Z}_p[G]$-**lattice** if

- M is a finitely generated left $\mathbb{Z}_p[G]$-module;
- M is a torsion-free $\mathbb{Z}_p$-module.

Definition

Let G be a profinite group, and let Ω be a profinite left G-set. Then $M = \mathbb{Z}_p[\![\Omega]\!]$ is called a left $\mathbb{Z}_p[\![G]\!]$-**permutation module**. If Ω is a transitive profinite left G-set, then the $\mathbb{Z}_p[\![G]\!]$-permutation module $\mathbb{Z}_p[\![\Omega]\!]$ will be also called **transitive**.

Al Weiss' result

Theorem (A. Weiss (1988))

Let G be a finite p-group, let N be a normal subgroup of G, and let M be a left $\mathbb{Z}_p[G]$-lattice such that

- $\mathrm{res}^G_N(M)$ *is a projective* $\mathbb{Z}_p[N]$*-module;*
- $M^N = \mathrm{Hom}_N(\mathbb{Z}_p, M)$ *is* $\mathbb{Z}_p[G/N]$*-permutation module.*

Then M is a left $\mathbb{Z}_p[G]$-permutation module.

Transitive permutation modules for pro-p groups

Theorem (T.W. & P. Zalesskii (2013))

Let G be a pro-p group, let N be a closed normal subgroup of G, and let M be a profinite left $\mathbb{Z}_p[\![G]\!]$-module with the following properties:

- *M_U is a torsion-free abelian pro-p group for every open, normal subgroup U of G; and*
- $\mathrm{res}^G_N(M) \simeq \mathbb{Z}_p[\![N]\!]$.

Then M is a transitive $\mathbb{Z}_p[\![G]\!]$-permutation module. In particular, there exists a closed subgroup C of G which is an N-complement, i.e., $G = C \cdot N$ and $C \cap N = \{1\}$.

The Herfort-Zalesskii theorem

without Bass-Serre theory ...

Theorem (W. Herfort, P. Zalesskii (2013))

Let G be a finitely generated pro-p group. Then the following are equivalent.

- *G is virtually free;*
- $\mathrm{vcd}_p(G) \leq 1$;
- *$G \simeq \pi_1(\mathcal{A}, x_0)^{\vee}_p$ for some finite graph of finite p-groups $\mathcal{A}$ s.th. $\pi_1(\mathcal{A}, x_0)$ is a residually finite p-group, and $\_^{\vee}_p$ denotes pro-p completion.*

Virtually free pro-p products

Theorem (T.W. & P. Zalesskii (2013))

Let G be a finitely generated pro-p group containing an open subgroup $H \simeq A \amalg B$ with $A, B \neq \{1\}$. Then G is isomorphic to the pro-p fundamental group of a finite graph of pro-p groups with finite edge stabilizers.

Low-dimensional group theory

	discrete groups	pro-p groups
$\mathrm{cd}(G) = 1$ $\Leftrightarrow G$ free	Stallings, Swan (1968), (1969)	folklore (Serre, 1962)
Stallings' dec.	Stallings (1971)	W. & Zalesskii (2013) Wingberg (2013)
virt. free groups	Karass & Pietrowski & Solitar (1973)	Herfort & Zalesskii (2013)
PD^2-groups	Eckmann & Müller (1980)	Demush'kin, Labute (1965)
1-relator groups	Lyndon (1950)	(??) (Labute)
surface grp. conj.	(??) (Rosenberger et al.)	Dummit & Labute (1983)
PD^3 & FAb groups (Thurston)	(??) (Perelman) (Wise, Agol, et al.)	(???)
Elem. type conj.	(???)	(???)