F a field, L a Lie algebra /F

$L = \langle V \rangle$, $\dim_F V < \infty$

$g_V(n) = \dim_F (V + \dots + V^n)$

Polynomial growth : $\exists$ polynomial

$p(t)$ $\quad g_V(n) \leq p(n)$

$$\min \deg p(t) = \limsup_{n \to \infty} \frac{\ln g_V(n)}{\ln n} =$$

<u>Gelfand-Kirillov</u> dimension of L.

<u>Part I</u>. char $F = p > 0$

<u>Part II</u>. char $F = 0$.

What is a Lie (associative) analog of Grigorchuk, Gupta-Sidki, etc groups?

Differential operators in infinitely many variables (corresponding to vertices)

<u>V. Petrogradsky</u>, 2004

char $F = 2$, $F[t_0, t_1, \dots \mid t_i^2 = 0]$

$\partial / \partial t_i = \partial_i$

$v_1 = \partial_1 + t_0 \partial_2 + t_0 t_1 \partial_3 + \dots t_0 \cdots t_{n-2} \partial_n + \cdots$

v_n (shift) $= \partial_n + t_{n-1} \partial_{n+1} + \cdots$

$L = \langle v_n, n \geq 1 \rangle = \langle v_1, v_2 \rangle$, nil: $a^{p^{n(a)}} = 0$

GKdim $L = \log_{\frac{1+\sqrt{5}}{2}} 2 \sim 1.4.$

<u>Shestakov-Z.</u>, 2005

char $F = p > 0$

$v_1 = \sum (t_0 \cdots t_{n-2})^{p-1} \partial_n$

$L = \langle v_n, n \geq 1 \rangle = \langle v_1, v_2 \rangle$

Criterion of nilness

$$\mathrm{GKdim}\, L = \log_{\frac{1+\sqrt{4p-3}}{2}} p, \qquad (1,2)$$

$$A = \mathrm{Assoc}\langle v_1, v_2 \rangle$$

$$\mathrm{GKdim}\, A \leq 2 \log_{\frac{1+\sqrt{4p-3}}{2}} p$$

<u>Motivation</u>

<u>Problem</u> : ? $\exists$ an associative nil algebra of polynomial growth

<u>Lenagan-Smoktunowicz</u> , 2008 : examples over countable fields

For uncountable fields , <u>Bell-Young</u> : subexponential growth

Is $A = \mathrm{Assoc}\langle v_1, v_2 \rangle$ nil ?

A is $\mathbb{Z}\times\mathbb{Z}$-graded by degrees in v_1, v_2

If $p \neq n^2+n+1$ then $A_{(0,0)} = (0)$

and A is graded nil.

If $p = n^2+n+1$ then $v_1 v_2$ is not

nilpotent (<u>Bartholdi</u>, <u>Krylюk</u>)

If $p \neq n^2+n+1$, is A nil?

Probably not (computer experiment,

<u>Bartholdi</u>)

<u>S. Sidki</u> : associative residually

finite algebra $A \hookrightarrow M_2(A)$,

finite codimension

<u>L. Bartholdi</u> : self-similar Lie

algebras

R finitely generated commutative algebra L,

$$L \to L \otimes_F R + \operatorname{Der} R$$

image of finite codimension.

In addition to our algebras: L(G) of Grigorchuk and Gupta-Sidki groups.

Nilness criterion $\Rightarrow$ L(G) of these groups are nil algebras.

<u>Grigorchuk</u>: $L(G) = L_1 + L_2 + \cdots$,

$\dim L_i \leq 2$

<u>Remark</u> $L(G) \otimes_{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}(t)$ is not nil.

Problems

(1) reconstruct self-similar groups from self-similar algebras

(2) vertex operators ...

Part 2 char $F = 0$.

No such things

E. Martinez - Z., 2001 : $L = L_1 + L_2 + \cdots$,

$\dim_F L_i \le d$, graded nil $\Rightarrow$ nilpotent

$L = \bigoplus_{i \in \mathbb{Z}} L_i$ graded simple

V. Kac, 67; O. Mathieu, 91 : L graded simple of polynomial growth iff isomorphic to one of the following algebras :

1) $\mathfrak{g}[t^{-1}, t]$, $\mathfrak{g}$ simple finite dimensional algebra

2) $W_n = \mathrm{Der}\, F[t_1, \ldots, t_m]$, S, H, K

3) $Vir = \mathrm{Der}\, F[t^{-1}, t]$

Superalgebras

Associative superalgebra $A = A_{\bar{0}} + A_{\bar{1}}$ $\quad \mathbb{Z}/2\mathbb{Z}$ - graded algebra

Lie superalgebra $L = L_{\bar{0}} + L_{\bar{1}} \subseteq A_{\bar{0}} + A_{\bar{1}}$, $[a_i, b_j] = a_i b_j - (-1)^{ij} b_j a_i$

70s : Superextensions of the Virasoro, $Vir \subseteq L_{\bar{0}}$

Nevieu, Shwartz, Ramon, Italyan physisists, Seiberg:

Superconformal algebras, so_N, su_2

Kac, Van de Leur, 88: $L = \bigoplus_{i \in \mathbb{Z}} L_i$,

graded simple, $\dim_F L_i \leq d$,

$Vir \subseteq L_{\bar{0}}$

1) $W(n:1) = \mathrm{Der}\, F[t^{-1}, t, \xi_1, \dots, \xi_m]$

2) $S(n, \alpha) = \{ w \in W(n:1), \dim(t^{\alpha} w) = 0 \}$

3) $K(n)$ contact brackets

Grozman, Leites, Shchepochkina;

Cheng-Kac, 1996:

4) $CK(6)$

Conjecture: these are all

Representation Theory: Graded Modules

$V = \oplus V_i$, $\dim L_i < \infty$

O. Mathieu, 92: irreducible graded modules over Vir

$V = \sum_{i=-\infty}^{N} V_i$ or $V = \sum_{i=N}^{\infty} V_i$ or

V = an intermediate module of

Feigin - Fuchs $V = F[t^{-1}, t]$; $\alpha, \beta \in F$

$$\left(f(t)\frac{d}{dt}\right) g(t) = -fg' + \alpha f' g + \beta \frac{1}{t} fg$$

Formal Distributions

$$a(z) = \sum_{i \in \mathbb{Z}} (t^i \xi_1 \cdots \xi_{i_n}) z^{-i-1}$$

Locality axiom: $[a(z), b(w)](z-w)^N = 0$.

vertex operator algebras,

conformal algebras

Equivalent to:

$L = \sum_i L_i$, $\dim L_i = d$

$e_{i1}, \dots, e_{id}$ basis in L_i

$$[e_{ip}, e_{jq}] = \sum_{k=1}^{d} \gamma_{pqk}(i,j)\, e_{i+j,k}$$

$\gamma_{pqk} : \mathbb{Z}^2 \to F$ d^3 functions

Locality Axiom $\Leftrightarrow$ all γ_{pqk} are polynomial

Last 20 years, <u>Kac</u> + his students:

description of irreducible conformal modules

For $CK(6)$ <u>Martinez-Z.</u> (to appear)

<u>Theorem</u> (Martinez-Z.) Except for $W(1:1)$ and Neveu-Schwartz graded irreducible modules $\xleftrightarrow{1-1}$ irreducible conformal modules.