LARGE ODD ORDER GROUPS OF FIXED SYMMETRIC GENUS

PRELIMINARY REPORT

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For odd order groups, the symmetric genus and the strong symmetric genus are the same.

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There are 10 odd order groups with genus between 2 and 26. The genus spectrum in this range is 7, 10, 12, 19, 21, 25 and 26.

We are also interested in the density of this genus spectrum in the positive integers.

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Proposition 1: **May & Zimmerman, 2008.** Let G be a finite group of odd order that acts on a Riemann surface X of genus $g \geq 2$. If $|G| > 8(g-1)$, then G has the given order and is a quotient of one of the following four triangle groups. We call these groups LO1 to LO4.

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What restrictions are there on the orders of groups in these classes?

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Therefore $H_{3p^2} \cong (Z_p \times Z_p) \times_\phi Z_3$.

Proposition: H_{3p^2} is a quotient of the triangle group $\Gamma(3,3,p)$.

Background Lemmas

Lemma 1: **May & Zimmerman, 2008.** Let $p \geq 5$ be a prime number and let $\Gamma = \Gamma(3,3,p)$. Then

(1) $\Gamma/\Gamma' \cong Z_3$.

(2) $\Gamma'/\Gamma'' \cong Z_p \times Z_p$.

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Proposition 5: Let G be an LO1 group. Then
(1) $G/G' \cong Z_3$.
(2) $G'/G'' \cong Z_5 \times Z_5$.
(3) $G/G'' \cong H_{75}$.

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Sketch of Proof: Let G be an LO1-group. Let J be a quotient of G of smallest order satisfying q divides $|J|$ and J is an LO1-group.

Let M be a minimal normal subgroup of J containing an element of order q. So $M \subseteq J''$.

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Suppose that $[J : L] = [H : K] = 3$.

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Therefore, $[H:K] > 3$ and so $\frac{H}{K}$ is an LO1-group. **May & Zimmerman, 2008.**

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The genus spectrum of LO1 groups has density zero in the positive integers.

Previous Theorems

May & Zimmerman, 2008.

Theorem

Let G_1 be an LO1-group that acts on the Riemann surface X_1 of genus $g_1 \geq 2$, with $|G_1| = 15(g_1 - 1)$ (so that the action is a genus action unless $G_1 \cong H_{75}$). Let $\phi : X \to X_1$ be a full covering of X_1 of degree $r > 1$. If the degree r is odd, then there is an LO1-group G with genus action on the surface X.

Previous Theorems 2

Corollary

If G is an LO1-group of order $15(g-1)$ and n is an odd positive integer, then G has an LO1-extension G_n of the form

$$1 \to (Z_n)^{2g} \to G_n \to G \to 1.$$

LO2-groups

Proposition

Let G be an LO2-group. Then G/G'' is isomorphic to either $G_{21} \cong Z_7 \times_\phi Z_3$ or H_{147}. In either case, G has a normal subgroup M such that $G/M \cong G_{21}$.

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Proposition

The groups G_{21} and H_{147} are the only LO2-groups with $\sigma = 1$. If G is an LO2-group with $|G| > 147$, then $\sigma(G) = 1 + 2|G|/21$ and G has odd genus. Further, $\sigma(G) \equiv 3 \ mod \ 4$.

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LO3-groups

LO3-groups are very different from the LO1, LO2 and LO4 groups because they are quotients of the triangle group $\Gamma(3,3,9)$ and the third period is not a prime. Nilpotent groups of this type have already been studied **Zomorrodian, 1987.**

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Question: Are there LO3 groups that are not 3-groups?

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Question: Are there LO3 groups that are not 3-groups?

It is easy to find such groups in the Magma Library of Groups. For example: SG(567, 17) is an LO3-group of order $3^4 \cdot 7$.

Simple Results

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Let $u = [y^{-1}, x^{-1}], v = [x, y^{-1}], w = [y, x^{-1}]$ and $z = w^{x^{-1}}$.

Now $\Gamma' = \langle u, v, w, z\rangle$ and the group Γ' has relators

$$(vw)^3 = (uz)^3 = (uwzv)^3 = 1.$$

Simple Results 2

Theorem

Let Γ *be the free* $(3,3,9)$ *triangle group.*
(1) $\Gamma/\Gamma' \cong Z_3 \times Z_3$.
(2) $\Gamma'/\Gamma'' \cong Z_3 \times Z_3 \times Z \times Z$.
(3) The actions of x *and* y *on* $\langle u, v, w, z\rangle$ *are easy to compute.*

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(3) The actions of x and y on $\langle u, v, w, z\rangle$ are easy to compute.

There are no finite metabelian LO3 groups that are also LO1 groups. An LO1 group has H_{75} as the metabelian quotient and it is not an LO3 group.

Finite Metabelian Quotients

Definition: Let $G(m)$ be the free metabelian $(3,3,9)$ triangle group with the added relator $([y^{-1},x^{-1}])^m = 1$.

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It is easy to show that this implies that all generators u, v, w and z, of G' have order m.

The group $G(m)$

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Therefore, we will consider the groups $G(3n)$.

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Theorem

Let G be a finite metabelian LO3-group, with $\sigma(G) \geq 2$. Then G is a quotient of $G(3n)$ for some integer n.

Sketch of Proof: The order of any of the commutators, u, v, w and z in G is $3n$ and so G is clearly a quotient of $G(3n)$.

Properties of $G(3n)$

Let $p > 3$ be a prime number. Let P be a Sylow p-subgroup of $G(3n)$. Since P is contained in $(G(3n))'$, it is an abelian p-group of rank 2 and is normal in $G(3n)$.

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Any finite metabelian LO3-group G, with $\sigma(G) \geq 2$ has Sylow p-subgroups that are abelian of rank 2 or less.

Theorem

Consider the group $G(3n)$ for some integer n.
(1) if 3 does not divide n, then $Z(G(3n)) \cong Z_3$, and
(2) if 3 divides n, then $Z(G(3n)) \cong Z_3 \times Z_3$.

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Corollary

Let G be a finite metabelian LO3-group, with $\sigma(G) \geq 2$ and let p be a prime, $p > 3$. If p divides $|G|$ and $p \equiv 2$ (mod 3), then p^2 divides $|G|$.

Sketch of Proof

G is a quotient of $G(3n)$, where $3n = o(u)$ and p divides n, but p^2 does not. So p^2 divides $|G(3n)|$ and p^3 does not.

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Let P be the Sylow p-subgroup of N and it is normal in $G(3n)$.

Now $|P| = p$ and so it is cyclic.

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Also $G(3n)$ acts non-trivially on P, since otherwise $P \subseteq Z(G(3n))$.

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Restrictions on LO3 groups

Theorem

Let G be a finite metabelian LO3-group, with $|G|$ divisible by 27. If $|G| = p^k m$ for some prime $p > 3$, where $\gcd(p, m) = 1$ and k is odd, then 3 divides $(p-1)$.

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Now G is a quotient of $G(n)$ by a normal subgroup N. Let Q be the Sylow p-subgroup of $G(n)$.

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Now some power of $Q \cap N$ is a normal subgroup of $G(n)$ of order p.

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Corollary

Let G be a finite metabelian LO3-group, with $|G|$ divisible by 27 and let p be a prime congruent to 2 (mod 3). If $|G| = p^k m$, where $\gcd(p, m) = 1$, then k is even.

Abundance of LO3 groups

Theorem

let p be a prime congruent to 1 (mod 3). Let $G = G(n)$, with $n = p^k m$, where $\gcd(p, m) = 1$ and 3 divides m. There exist two normal subgroups N of order p so that G/N is an LO3-group.

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let p be a prime congruent to 1 (mod 3). Let $G = G(n)$, with $n = p^k m$, where $\gcd(p, m) = 1$ and 3 divides m. There exist two normal subgroups N of order p so that G/N is an LO3-group.

Proof: Let u and v be the power of the commutators defined in G, so that they have order p.

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There are two possible values of t satisfying $t^3 \equiv 1 \ (mod\ p)$ and t is not congruent to $1 \ (mod\ p)$. Notice that $t + 1 \equiv -t^2 \ (mod\ p)$.

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Let $G = G(n)$, where $n = 3m$, There exists a normal subgroup N of order 9 so that G/N is an LO3-group.

Symmetric Genus One

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Now M is one of the four groups $M_1 = \langle x \rangle G'$, $M_2 = \langle y \rangle G'$, $M_3 = \langle xy \rangle G'$ or $M_4 = \langle x^{-1} y \rangle G'$.

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These are SG(81,9) or SG(243,26).

Abundance of Metabelian LO3 groups

Theorem

Let R be a finite set of primes, $p \equiv 1 \pmod 3$ and S be a finite set of primes, $p \equiv 2 \pmod 3$. For each prime $p \in R$, let ℓ be a positive integer and for each prime $p \in S$, let $2k$ be a positive even integer. Finally choose an integer $q \geq 4$. Then there exists a metabelian LO3-group G of order $3^q \cdot \prod_{p \in R} p^\ell \cdot \prod_{p \in S} p^{2k}$.
Furthermore, $\sigma(G) = 1 + 3^{q-2} \cdot \prod_{p \in R} p^\ell \cdot \prod_{p \in S} p^{2k}$.

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We believe that this is best possible.

Possible Example

The group $SG(1053, 51)$ is an LO3 group with $|G/G'| = 3$, $|G'/G''| = 13$ and $G'' \cong Z_3^3$. So G/G'' is the unique non-abelian group of order 39, H_{39}.

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The semidirect product $SG(1053, 51) \times_\phi Z_{29}^3$ should be an LO3 group with the right properties.