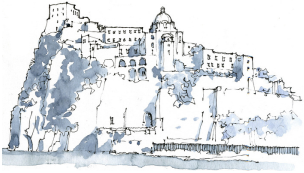
The Zassenhaus filtration and $\mathbb{F}_p$-cohomology for profinite groups



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To the memory of
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Introduction

How much information on the structure of a group G does the cohomology provide?

The following results shed some light on some cases (with arithmetic relevance) when the structure of the $\mathbb{F}_p$-cohomology of a group G determines the structure of a canonical algebra related to G.

These results have the following arithmetic motivation:

Koszulity conjecture (L. Positselski, 2015)

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Pro-p groups

Let p be a prime number.

Recall: a pro-p group G is a compact group s.t. the unit element has a basis of neighbourhoods consisting of open and closed normal subgroups of index a p-power.

Galois groups of (infinite) Galois p-extensions are pro-p groups. Arithmetic people are greatly interested in maximal pro-p quotients and pro-p Sylow subgroups of Absolute Galois groups of fields.

The p-Zassenhaus filtration of a group

For a group G the p-Zassenhaus filtration $(G_{(n)})_{n\geq 1}$ is the fastest descending series such that $G_{(n)}^p \subseteq G_{(np)}$ and $[G_{(i)}, G_{(j)}] \subseteq G_{i+j}$

$$\text{One has} \quad G_{(n)} = G_{\lceil n/p \rceil} \cdot \prod_{i+j=n} [G_{(i)}, G_{(j)}] = \prod_{ip^h \geq p} \gamma_i(G)^{p^h}$$

Equivalently, one has $G_{(n)} = \{g \in G \mid g - 1 \in I^n\}$, where $I = \langle g - 1 \rangle$ is the augmentation ideal of the group algebra $\mathbb{F}_p[G]$

We define the (non-negatively) graded algebra over $\mathbb{F}_p$

$$\mathrm{gr}_\bullet(G) = \bigoplus_{n\geq 0} I^n/I^{n+1}, \quad I^0 = \mathbb{F}_p[G],$$

so that $\mathrm{gr}_0(G) = \mathbb{F}_p$ and $G_{(n)}/G_{(n+1)} \hookrightarrow \mathrm{gr}_n(G)$ for all $n \geq 1$.

The graded group algebra and $\mathbb{F}_p$-cohomology

If F is a free pro-p group on d generators, then $\mathrm{gr}_\bullet(F)$ is the free associative algebra $\mathbb{F}_p\langle X\rangle$, with $X = \{X_1, \ldots, X_d\}$, and the elements of $F_{(n)}/F_{(n+1)}$ are homogeneous polynomials of degree n in $\mathbb{F}_p\langle X\rangle$.

In general, if $G = F/R$ then $\mathrm{gr}_\bullet(G)$ is a quotient of $\mathrm{gr}_\bullet(F) = \mathbb{F}_p\langle X\rangle$.

$\mathbb{F}_p$-cohomology

The cohomology with coefficients in $\mathbb{F}_p$ is endowed with the cup product

$$\cup\colon H^r(G, \mathbb{F}_p) \times H^s(G, \mathbb{F}_p) \longrightarrow H^{r+s}(G, \mathbb{F}_p)$$

which is graded-commutative, i.e., $\beta \cup \alpha = (-1)^{rs}\alpha \cup \beta$.

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The cup product and defining relations

How do these two things relate?

$$\text{Let}\qquad 1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

be a minimal presentation ($\Rightarrow R \subseteq F_{(2)}$) with R generated by $r_1, \ldots, r_m$ as normal subgroup of F.

Theorem

For a finitely generated pro-p group G the following are equivalent

- $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \twoheadrightarrow H^2(G, \mathbb{F}_p)$;
- $R \cap F_{(3)} = R^p[R, F]$;
- $r_i \in F_{(2)} \smallsetminus F_{(3)}$ for all i – i.e., the images of the r_i's in $\mathrm{gr}_\bullet(F) = \mathbb{F}_p\langle X\rangle$ are of degree 2.

Quadratic algebras

A graded algebra $A_\bullet = \bigoplus_{n\geq 0} A_n$, $A_0 = \mathbb{F}_p$, is quadratic if $A_\bullet = \mathbb{F}_p\langle X\rangle/\langle\Omega\rangle$, Ω a set of homogeneous polynomials of degree 2, with $X = \{X_1, \ldots, X_d\}$.

Examples

- $S_\bullet(X) = \mathbb{F}_p\langle X\rangle/\langle\Omega\rangle$ with $\Omega = \{X_iX_j - X_jX_i, i,j \in \ldots\}$;
- $\Lambda_\bullet(X) = \mathbb{F}_p\langle X\rangle/\langle\Omega\rangle$ with $\Omega = \{X_iX_j + X_jX_i, i,j \in \ldots\}$.

The quadratic dual of a quadratic algebra $A_\bullet = \mathbb{F}_p\langle X\rangle/\langle\Omega\rangle$ is the algebra $A_\bullet^! = \mathbb{F}_p\langle X^*\rangle/\langle\Omega^\perp\rangle$, with

$$\Omega^\perp = \{\phi \in (\mathbb{F}_pX^*)^{\otimes 2} \mid \phi(f) = 0 \; \forall\, f \in \Omega\}$$

and X^* a dual basis of X – i.e., $X_i^*(X_j) = \delta_{ij}$.

E.g., $S_\bullet(X)^! = \Lambda_\bullet(X^*)$, as $(X_i^*X_j^* + X_j^*X_i^*)(X_hX_k - X_kX_h) = 0$ for all i, j, h, k.

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Quadratic cohomology and the algebra $\mathrm{gr}_\bullet(G)$

Let $1 \to R \to F \to G \to 1$ be again a minimal presentation.

Theorem

If G is a finitely generated pro-p group and the cohomology $H^\bullet(G, \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(G, \mathbb{F}_p)$ is quadratic then

- $H^\bullet(G, \mathbb{F}_p)^! = \mathbb{F}_p\langle X \rangle / \mathcal{R}$, with $\mathcal{R}$ the ideal generated by the images of the r_i's in $\mathbb{F}_p\langle X \rangle$ (they are of degree 2);
- $H^\bullet(G, \mathbb{F}_p)^! \twoheadrightarrow \mathrm{gr}_\bullet(G)$ (it is an isomorphism in degree 0, 1, 2), and if $\mathrm{gr}_\bullet(G)$ is quadratic then we have an isomorphism.

Pro-p Sylow subgroups of absolute Galois groups of fields, and Galois groups of certain p-extensions of number fields with restricted ramification have quadratic $\mathbb{F}_p$-cohomology.

Example 1: 1-relator pro-p groups

Let G be a one-relator group, i.e., $R = \langle r \rangle$. If $r \in F_{(2)} \smallsetminus F_{(3)}$ (r has "degree" 2) then there is a basis $X = \{X_1, \ldots, X_d\}$ such that

$$\bar{r} = [X_1, X_2] + [X_3, X_4] + \ldots + [X_{s-1}, X_s] \in F_{(2)}/F_{(3)} \subseteq \mathbb{F}_p\langle X \rangle$$

for some even $s \leq d$ (here $[X_i, X_j] = X_iX_j - X_jX_i$). Also

$$H^\bullet(G, \mathbb{F}_p) = \mathbb{F}_p \oplus \left(\bigoplus_{i=1}^{d} \mathbb{F}_p.X_i \right) \oplus \mathbb{F}_p.X_1^*X_2^*, \qquad \text{and}$$

$$\mathrm{gr}_\bullet(G) = \mathbb{F}_p\langle X \rangle / \langle \bar{r} \rangle,$$

where $X_{2i-1}^*X_{2i}^* = -X_{2i}^*X_{2i-1}^* = X_1^*X_2^*$, for $i = 1, \ldots, s/2$, and all other monomials of degree 2 are 0.

Example 2: Right angled Artin pro-p groups

Set $G = F/R$ with $R = \{[x_i, x_j], 1 \leq i < j \leq d\}$ s.t. if $[x_i, x_j], [x_j, x_k] \in R$ then $[x_i, x_k] \notin R$ (no "triangles").

E.g.: $$\begin{array}{ccccc} x_1 & — & x_2 & & x_3 \\ | & & | & & | \\ x_4 & — & x_5 & — & x_6 \end{array} \quad \rightsquigarrow \quad R = \left\langle \begin{array}{c} [x_1, x_2], [x_2, x_5] \\ [x_5, x_4], [x_4, x_1] \\ [x_5, x_6], [x_6, x_3] \end{array} \right\rangle$$

Then

$$H^\bullet(G, \mathbb{F}_p) = \mathbb{F}_p \oplus \left(\bigoplus_{i=1}^{d} \mathbb{F}_p . X_i \right) \oplus \left(\bigoplus_{[x_i, x_j] \in R} \mathbb{F}_p . X_i^* X_j^* \right), \quad \text{and}$$

$$\mathrm{gr}_\bullet(G) = \mathbb{F}_p \langle X \rangle / \langle X_i X_j - X_j X_i \rangle, \quad \text{s.t. } [x_i, x_j] \in R$$

(In $H^\bullet(G, \mathbb{F}_p)$ one has $X_j^* X_i^* = -X_i^* X_j^*$ and all other degree-2 monomials are 0.)

Example 3: free products

The graded algebras $H^\bullet(G,\mathbb{F}_p)$ and $\mathrm{gr}_\bullet(G)$ "behave well" with respect to free products (in the category of pro-p groups):

If $G = G_1 * G_2$ then

$$H^\bullet(G,\mathbb{F}_p) = H^\bullet(G_1,\mathbb{F}_p) \sqcap H^\bullet(G_2,\mathbb{F}_p), \quad \text{and}$$
$$\mathrm{gr}_\bullet(G) = \mathrm{gr}_\bullet(G_1) \sqcup \mathrm{gr}_\bullet(G_2).$$

Moreover, if $H^\bullet(G_i,\mathbb{F}_p)^! \simeq \mathrm{gr}_\bullet(G_i)$ for $i = 1,2$, then also

$$H^\bullet(G,\mathbb{F}_p)^! \xrightarrow{\sim} \mathrm{gr}_\bullet(G)\ .$$

THANKS I

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