Representation Growth of Arithmetic Groups

Michele Zordan

University of Bielefeld

michele.zordan@math.uni-bielefeld.de

April 2, 2016
Definition
Let $G$ be a group. For $n \in \mathbb{N}$, we denote by $r_n(G)$ the number of isomorphism classes of $n$-dimensional irreducible complex representations of $G$. 
Representation growth function

Definition

Let $G$ be a group. For $n \in \mathbb{N}$, we denote by $r_n(G)$ the number of isomorphism classes of $n$-dimensional irreducible complex representations of $G$.

When $G$ is a topological or an algebraic group, it is tacitly understood that representations enumerated by $r_n(G)$ are continuous or rational, respectively.

Definition

We say that $G$ is (representation) rigid when $r_n(G)$ is finite for all $n \in \mathbb{N}$. 
Representation growth function

**Definition**
Let $G$ be a group. For $n \in \mathbb{N}$, we denote by $r_n(G)$ the number of isomorphism classes of $n$-dimensional irreducible complex representations of $G$.

When $G$ is a topological or an algebraic group, it is tacitly understood that representations enumerated by $r_n(G)$ are continuous or rational, respectively.

**Definition**
We say that $G$ is (representation) *rigid* when $r_n(G)$ is finite for all $n \in \mathbb{N}$. 
The function $r_n(G)$ as $n$ varies in $\mathbb{N}$ is called the representation growth function of $G$. 

**PRG**
The function $r_n(G)$ as $n$ varies in $\mathbb{N}$ is called the representation growth function of $G$.

**Definition**

If the sequence

$$R_N(G) = \sum_{n=1}^{N} r_n(G) \text{ for } N \in \mathbb{N},$$

is bounded by a polynomial in $N$, the group $G$ is said to have *polynomial representation growth* (PRG).
The representation growth of a rigid group can be studied by means of the *representation zeta function*, namely, the Dirichlet series

\[
ζ_G(s) = \sum_{n=1}^{∞} r_n(G)n^{-s},
\]

where \(s\) is a complex variable.
Abscissa of convergence

Definition
The abscissa of convergence \( \alpha(G) \) of the series \( \zeta_G(s) \) is the infimum of all \( \alpha \in \mathbb{R} \) such that \( \zeta_G(s) \) converges on the complex half-plane \( \{ s \in \mathbb{C} \mid \Re(s) > \alpha \} \)
Abscissa of convergence

**Definition**

The *abscissa of convergence* $\alpha(G)$ of the series $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges on the complex half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$
Abscissa of convergence

Definition

The absissa of convergence $\alpha(G)$ of the series $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that $\zeta_G(s)$ converges on the complex half-plane $\{s \in \mathbb{C} \mid \Re(s) > \alpha\}$

Proposition

Let $G$ have PRG. The absissa of convergence $\alpha(G)$ is the smallest value such that

$$R_N(G) = O(1 + N^{\alpha(G) + \varepsilon})$$

for every $\varepsilon \in \mathbb{R}_{>0}$
Larsen and Lubotzky conjecture

Larsen and Lubotzky made the following conjecture.

Conjecture (Larsen and Lubotzky, 2008)

Let $H$ be a higher-rank semisimple group. Then, for any two irreducible lattices $\Gamma_1$ and $\Gamma_2$ in $H$, $\alpha(\Gamma_1) = \alpha(\Gamma_2)$. 

• In 2011 Avni, Klopsch, Onn and Voll proved a variant of Larsen and Lubotzky conjecture for higher-rank semisimple groups in characteristic 0 assuming that both $\alpha(\Gamma_1)$ and $\alpha(\Gamma_2)$ are finite.

• Using $p$-adic integration and approximative Clifford theory, the same authors proved Larsen and Lubotzky’s conjecture for groups of type $A_2$. 

Main results

$p$-adic Lie theory
Zeta function as product of geometric progressions
The representation zeta function of $SL_4(\alpha)$
Larsen and Lubotzky conjecture

Larsen and Lubotzky made the following conjecture.

**Conjecture (Larsen and Lubotzky, 2008)**

Let $H$ be a higher-rank semisimple group. Then, for any two irreducible lattices $\Gamma_1$ and $\Gamma_2$ in $H$, $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.

- In 2011 Avni, Klopsch, Onn and Voll proved a variant of Larsen and Lubotzky conjecture for higher-rank semisimple groups in characteristic 0 assuming that both $\alpha(\Gamma_1)$ and $\alpha(\Gamma_2)$ are finite.
Larsen and Lubotzky conjecture

Larsen and Lubotzky made the following conjecture.

Conjecture (Larsen and Lubotzky, 2008)

Let $H$ be a higher-rank semisimple group. Then, for any two irreducible lattices $\Gamma_1$ and $\Gamma_2$ in $H$, $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.

- In 2011 Avni, Klopsch, Onn and Voll proved a variant of Larsen and Lubotzky conjecture for higher-rank semisimple groups in characteristic 0 assuming that both $\alpha(\Gamma_1)$ and $\alpha(\Gamma_2)$ are finite.

- Using $p$-adic integration and approximative Clifford theory, the same authors proved Larsen and Lubotzky’s conjecture for groups of type $A_2$. 
Definition

An arithmetic group is a group $\Gamma$ which is commensurable to $H(O)$, where $H$ is a connected, simply connected semisimple linear algebraic group defined over a number field $k$ and $O$ is the ring of integers in $k$. 
**Definition**

An arithmetic group is a group $\Gamma$ which is commensurable to $H(O)$, where $H$ is a connected, simply connected semisimple linear algebraic group defined over a number field $k$ and $O$ is the ring of integers in $k$.

We make the following simplification: from now on an arithmetic group is $H(O)$ for $H$ and $O$ as above.
Definition
Let $\Gamma = H(\mathcal{O})$ be an arithmetic group with $\mathcal{O}$ as above and $H \leq \text{GL}_d$ for some $d \in \mathbb{N}$. A principal congruence subgroup of level $m$ of $\Gamma$ is $\Gamma \cap l_d + \text{Mat}_d(p^m)$ for $p$ a prime ideal in $\mathcal{O}$.
Congruence subgroups

Definition
Let $\Gamma = H(\mathcal{O})$ be an arithmetic group with and $\mathcal{O}$ as above and $H \leq \mathrm{GL}_d$ for some $d \in \mathbb{N}$. A principal congruence subgroup of level $m$ of $\Gamma$ is $\Gamma \cap I_d + \mathrm{Mat}_d(p^m)$ for $p$ a prime ideal in $\mathcal{O}$.

Definition (Congruence subgroup)
A subgroup of an arithmetic group $\Gamma$ is called a congruence subgroup when it contains a principal congruence subgroup.
Definition (Congruence subgroup property)

Let $S$ be the set of archimedean places of $\mathcal{O}$. We say that an arithmetic group $\Gamma = H(\mathcal{O})$ has the weak congruence subgroup property (wCSP) when the map

$$\hat{H}(\mathcal{O}) \to H(\hat{\mathcal{O}})$$

has finite kernel.
Definition (Congruence subgroup property)

Let $S$ be the set of archimedean places of $\mathcal{O}$. We say that an arithmetic group $\Gamma = H(\mathcal{O})$ has the weak congruence subgroup property (wCSP) when the map

$$\hat{H}(\mathcal{O}) \to H(\hat{\mathcal{O}})$$

has finite kernel.

Theorem (Lubotzky and Martin, 2004)

Let $\Gamma$ be an arithmetic group in characteristic 0. Then $\Gamma$ has PRG if and only if it has the wCSP.
Euler products

Proposition (Larsen and Lubotzky 2008)

When $\Gamma$ has the CSP, the representation zeta function of $\Gamma$ admits an Euler product decomposition.
Euler products

Proposition (Larsen and Lubotzky 2008)
When $\Gamma$ has the CSP, the representation zeta function of $\Gamma$ admits an Euler product decomposition.

Let $\Gamma = H(\mathcal{O})$, and let $S$ be the set of archimedean places in $\mathcal{O}$. The Euler product decomposition is

$$\zeta_\Gamma(s) = \zeta_{H(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \cdot \prod_{v \notin S} \zeta_{H(\mathcal{O}_v)}(s).$$
Euler products

Proposition (Larsen and Lubotzky 2008)

*When $\Gamma$ has the CSP, the representation zeta function of $\Gamma$ admits an Euler product decomposition.*

Let $\Gamma = H(\mathcal{O})$, and let $S$ be the set of archimedean places in $\mathcal{O}$. The Euler product decomposition is

$$\zeta_{\Gamma}(s) = \zeta_{H(\mathbb{C})}(s)^{|k:\mathbb{Q}|} \cdot \prod_{v \notin S} \zeta_{H(\mathcal{O}_v)}(s).$$

- The first factor enumerates the *rational* irreducible representations of the group $H(\mathbb{C})$ and is known as Witten zeta function.
Euler products

Proposition (Larsen and Lubotzky 2008)

*When $\Gamma$ has the CSP, the representation zeta function of $\Gamma$ admits an Euler product decomposition.*

Let $\Gamma = H(\mathcal{O})$, and let $S$ be the set of archimedean places in $\mathcal{O}$. The Euler product decomposition is

$$\zeta_\Gamma(s) = \zeta_{H(\mathbb{C})}(s)^{|k: \mathbb{Q}|} \cdot \prod_{v \notin S} \zeta_{H(\mathcal{O}_v)}(s).$$

*• The first factor enumerates the rational irreducible representations of the group $H(\mathbb{C})$ and is known as Witten zeta function.*

*• The factors indexed by $v \notin S$ are representation zeta functions of compact $p$-adic analytic groups counting irreducible representations with finite image (i.e. continuous irreducible representations).*
Potent and saturable subgroups

Let $G$ be a connected simply connected semisimple linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Let $k$ be a number field with ring of integers $\mathcal{O}$ and completion $\mathfrak{o}$ with respect to a prime ideal $\mathfrak{p}$. We set $G = G(\mathfrak{o})$ and $\mathfrak{g} = \mathfrak{g}(\mathfrak{o})$. 
Potent and saturable subgroups

Let $G$ be a connected simply connected semisimple linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Let $k$ be a number field with ring of integers $\mathcal{O}$ and completion $\mathfrak{o}$ with respect to a prime ideal $\mathfrak{p}$. We set $G = G(\mathfrak{o})$ and $\mathfrak{g} = \mathfrak{g}(\mathfrak{o})$. The principal congruence subgroup of $G$ of level $m$ is

$$G^m = \ker(G \to G(\mathfrak{o}/\mathfrak{p}^m))$$
Potent and saturable subgroups

Let $G$ be a connected simply connected semisimple linear algebraic group defined over $\mathbb{Z}$ with Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Let $k$ be a number field with ring of integers $\mathcal{O}$ and completion $\mathfrak{o}$ with respect to a prime ideal $p$. We set $G = G(\mathfrak{o})$ and $\mathfrak{g} = \mathfrak{g}(\mathfrak{o})$.

The principal congruence subgroup of $G$ of level $m$ is

$$G^m = \ker(G \to G(\mathfrak{o}/p^m))$$

Proposition (Avni, Klopsch, Onn and Voll, 2013)

Let $e = e(\mathfrak{o}, \mathbb{Z}_p)$ be the absolute ramification index of $\mathfrak{o}$. If $m > e \cdot (p - 1)^{-1}$, then $G^m$ is saturable. Moreover, if $p > 2$ and $m \geq e \cdot (p - 2)^{-1}$, then $G^m$ is potent. If $p = 2$ and $m \geq 2e$, then $G^m$ is potent.
Let $\mathcal{L} = g(\mathbb{C})$ and let $d = \dim_{\mathbb{C}} \mathcal{L}$. We define the locus of constant centralizer dimension $k \leq d$

$$X^k_{\mathcal{L}}(\mathbb{C}) = \{x \in \mathcal{L} \mid \dim_{\mathbb{C}} C_{\mathcal{L}}(x) = k\}.$$ 

and we set

$$f_k = \dim_{\mathbb{C}} X^k_{\mathcal{L}}(\mathbb{C}),$$
Zeta function as product of geometric progressions

Theorem (MZ)

Let $S \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. 

Let $S \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. 

Zeta function as product of geometric progressions

Theorem (MZ)

Let $S \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $L$. Assume that the Killing form on $\mathfrak{g}$ is non-degenerate.
Zeta function as product of geometric progressions

**Theorem (MZ)**

Let $S \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. Assume that the Killing form on $\mathfrak{g}$ is non-degenerate. Assume further that $\mathfrak{g}$ has smooth and irreducible loci of constant centralizer dimension.
Zeta function as product of geometric progressions

**Theorem (MZ)**

Let $S \subseteq \{1, \ldots, d\}$ be the set of all possible dimensions for centralizers in $\mathcal{L}$. Assume that the Killing form on $\mathfrak{g}$ is non-degenerate. Assume further that $\mathfrak{g}$ has smooth and irreducible loci of constant centralizer dimension. Then for all $m \in \mathbb{N}$ such that $G^m$ is potent and saturable

$$\zeta_{G^m}(s) = q^{d \cdot m} \sum_{I \subseteq S} g_{\mathfrak{g}, I}(q) \cdot \prod_{i \in I} \frac{q^{f_i - (d - i) \frac{s + 2}{2}}}{1 - q^{f_i - (d - i) \frac{s + 2}{2}}}.$$
Let \( \mathfrak{o} \) be a compact discrete valuation ring of characteristic 0 whose residue field has cardinality \( q \) and characteristic not equal to 2. Then, for all \( m \in \mathbb{N} \) such that \( \text{SL}_4^m(\mathfrak{o}) \) is potent and saturable,

\[
\zeta_{\text{SL}_4^m(\mathfrak{o})}(s) = q^{15m} \frac{\mathcal{F}(q, q^{-s})}{\mathcal{G}(q, q^{-s})}
\]

where
\[ F(q, t) = qt^{18} - (q^7 + q^6 + q^5 + q^4 - q^3 - q^2 - q)t^{15} \\
+ (q^8 - 2q^5 - q^3 + q^2)t^{14} \\
+ (q^9 + 2q^8 + 2q^7 - 2q^5 - 4q^4 - 2q^3 - q^2 + 2q + 1)t^{13} \\
- (q^{10} + q^9 + q^8 - 2q^7 - 2q^6 - 2q^5 + 2q^3 + q^2 + q)t^{12} \\
+ (q^8 + 2q^6 + q^4 - q^3 - q^2 - q)t^{11} + (q^8 + q^7 - 2q^4 + q)t^{10} \\
- (2q^{10} + q^9 + q^8 - q^7 - 3q^6 - 2q^5 - 3q^4 - q^3 + q^2 + q + 2)t^9 \\
+ (q^9 - 2q^6 + q^3 + q^2)t^8 - (q^9 + q^8 + q^7 - q^6 - 2q^4 - q^2)t^7 \\
- (q^9 + q^8 + 2q^7 - 2q^5 - 2q^4 - 2q^3 + q^2 + q + 1)t^6 \\
+ (q^{10} + 2q^9 - q^8 - 2q^7 - 4q^6 - 2q^5 + 2q^3 + 2q^2 + q)t^5 \\
+ (q^8 - q^7 - 2q^5 + q^2)t^4 + (q^9 + q^8 + q^7 - q^6 - q^5 - q^4 - q^3)t^3. \]

\[ G(q, t) = q^9(1 - qt^3)(1 - qt^4)(1 - q^2t^5)(1 - q^3t^6). \]