Outer commutators in iterated wreath products of cyclic groups

Agnieszka Bier

Institute of Mathematics, Silesian University of Technology, Poland

agnieszka.bier@polsl.pl

Table representation of elements in $\wr_{i=1}^{n} C_p^{(i)}$

Let $P_n = \wr_{i=1}^{n} C_p^{(i)}$ denote the iterated wreath product of cyclic groups of order p.
P_n is known to be isomorphic to the Sylow p-subgroup of iterated wreath product of symmetric groups $\wr_{i=1}^{n} S_p^{(i)}$ and hence to the Sylow p-subgroup of the automorphism group of a p-adic rooted tree of height n [1].
Following [2] we represent elements of P_n as tables with n coordinates:

$$\mathbf{a} = [a_1, a_2(x_1), a_3(x_1, x_2), ..., a_n(x_1, ...x_{n-1})],$$

where a_1 is an element of the finite p-element field $\mathbb{F}_p$,
$a_k(x_1, ...x_{k-1}) = a_k(\underline{x}_{k-1}) \in \mathbb{F}_p[x_1, ..., x_{k-1}]/\mathcal{I}_{k-1}$,
and $\mathcal{I}_k = \langle x_1 - x_1^p, x_2 - x_2^p, ..., x_k - x_k^p\rangle$ is an ideal of the respective ring of polynomials of $k-1$ indeterminates over $\mathbb{F}_p$.
The table $\mathbf{a}$ consists of reduced polynomials over $\mathbb{F}_p$, with degrees in every indeterminate not greater than $p-1$. Since every function $f : \mathbb{F}_p^k \longrightarrow \mathbb{F}_p$ can be uniquely represented by a reduced polynomial $a_k(\underline{x}_k) \in \mathbb{F}_p[x_1, ..., x_k]/\mathcal{I}_k$, the table representation of elements in P_n is well defined and unique.

For every monomial $m(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) = c \cdot x_1^{\varepsilon_1} x_2^{\varepsilon_2} ... x_k^{\varepsilon_k}$ we define the height:

$$H(m(\varepsilon_1, \varepsilon_2, ..., \varepsilon_k)) = 1 + \varepsilon_1 + \varepsilon_2 p + ... + \varepsilon_k p^{k-1},$$

which is a natural number between 1 and $p^k - 1$. Additionally we set $H(0) = 0$.
If $f(x_1, ..., x_k) \in \mathbb{F}_p[x_1, ..., x_k]/\mathcal{I}_k$, then we define its height to be the maximal height of all its constituent monomials. So, every table $\mathbf{a} \in P_n$ may be characterized by a vector of heights of its coordinate polynomials:

$$char\ \mathbf{a} = [H(a_1), H(a_2(x_1)), ..., H(a_n(x_1, ..., x_{n-1}))],$$

which is called the characteristic of $\mathbf{a}$.

Group action on $\mathbb{F}_p^n$ and group operations

Let $\mathbf{a}, \mathbf{b} \in P_n$ be represented by two tables:

$$\mathbf{a} = [a_1, a_2(x_1), a_3(x_1, x_2), ..., a_n(x_1, ...x_{n-1})],$$

$$\mathbf{b} = [b_1, b_2(x_1), b_3(x_1, x_2), ..., b_n(x_1, ...x_{n-1})].$$

The group P_n acts on $\mathbb{F}_p^n$ as follows:

$$[x_1, ..., x_n]^{\mathbf{a}} = [x_1 + a_1, x_2 + a_2(x_1), ..., x_n + a_n(x_1, ...x_{n-1})].$$

Denoting by $\underline{x}_i = [x_1, ..., x_i]$ and $\mathbf{a}_i = [a_1, a_2(x_1), , ..., a_i(x_1, ...x_{i-1})]$ the group operations are given as:

$$\mathbf{a} \cdot \mathbf{b} = [a_1 + b_1, a_2(x_1) + b_2(\underline{x_1}^{a_1}), ..., a_n(\underline{x}_{n-1}) + b_n(\underline{x}_{n-1}^{\mathbf{a}_{n-1}})],$$

$$\mathbf{a}^{-1} = [-a_1, -a_2(\underline{x}_1^{\mathbf{a}_1^{-1}}), ..., -a_n(\underline{x}_{n-1}^{\mathbf{a}_{n-1}^{-1}})].$$

Observation [2]: $H\left(b_i(\underline{x}_{i-1}^{\mathbf{a}_i})\right) = H\left(b_i(\underline{x}_{i-1})\right)$ for any $\mathbf{a} \in P_n$.

Lower central series of P_n

A basic commutator of weight k is a word in alphabet $X = \{x_1, x_2, x_3, ...\}$ defined recursively as:

$$c_1(x_1) = x_1, \qquad c_k(x_1, ..., x_k) = [c_{k-1}(x_1, ..., x_{k-1}), x_k], \ k > 1.$$

Given group G, by $\gamma_k(G)$ we denote the k-th term of its lower central series, i.e. the subgroup generated by values of c_k:

$$\gamma_k(G) = \langle c_k(g_1, ..., g_k), \quad g_1, ..., g_k \in G\rangle.$$

The following characterization of the lower central series of P_n is elaborated by L. Kaloujninie in [2]:

$$\gamma_k(P_n) = \{\mathbf{a} \in P_n : \quad \forall i = 1, ..., n \quad H(\mathbf{a}_i) \le p^{i-1} - k + 1\}.$$

Basic commutators width

Every element $\mathbf{b} \in \gamma_k(P_n)$ is obviously a product of values of basic commutators:

$$\mathbf{b} = c_k(\mathbf{a}^{(1,1)}, \mathbf{a}^{(1,2)}, .., \mathbf{a}^{(1,k)}) \cdot c_k(\mathbf{a}^{(2,1)}, \mathbf{a}^{(2,2)}, .., \mathbf{a}^{(2,k)}) \cdots c_k(\mathbf{a}^{(s,1)}, \mathbf{a}^{(s,2)}, .., \mathbf{a}^{(s,k)})$$

for certain $s \in \mathbb{N}$ and $\mathbf{a}^{(i,j)} \in P_n$.

An interesting question is whether the number s can be estimated or bounded, or in particular, if $s = 1$. The following theorem provides the answer:

Theorem 1. *Every element of the lower central series term $\gamma_k(P_n)$ is a value of the basic commutator word c_k.*

<u>About the proof:</u>

- Induction on k
- Every element $\mathbf{b} \in \gamma_k(P_n)$ is a commutator of the form: $\mathbf{b} = [\mathbf{a}, \mathbf{u}]$, where

$$\mathbf{u} = [1, x_1^{p-1}, x_1^{p-1} x_2^{p-1}, ..., x_1^{p-1} x_2^{p-1} ... x_{n-1}^{p-1}].$$

- Tool: Kaloujnine's criterion on solvability of polynomial equations:

$$a_i(\underline{x}_{i-1}) - a_i(\underline{x}_{i-1}^{\mathbf{u}_{i-1}}) = b_i(\underline{x}_{i-1})$$

  The equation has a solution $a_i(\underline{x}_{i-1})$ iff the sum of values of $b_i(\underline{x}_{i-1})$ over each orbit of the action of $\mathbf{u}_{i-1}$ on $\mathbb{F}_p^{i-1}$ is zero.
- $\mathbf{u}$ is an element with the maximal characteristic in P_n and has only one orbit on $\mathbb{F}_p$.
- $\mathbf{b} \in \gamma_2(P_n)$, hence its characteristic on $i-th$ coordinate does not exceed $p^i - 1$
- every monomial in $b_i(\underline{x}_{i-1})$ contains at least one indeterminate in exponent less than $p-1$
- sum of all values of $b_i(\underline{x}_{i-1})$ over $\mathbb{F}_p^{i-1}$ is zero

Corollary 2. *Every element of the lower central series term $\gamma_k(P_n)$ is a value of the Engel word $e_k(x, y) = [x, \underbrace{y, ..., y}_{k-1}]$.*

Outer commutators

An outer commutator of weight k is any word in alphabet $X = \{x_1, x_2, x_3, ...\}$ of the form defined by:

$$\omega_1(x_1) = x_1, \qquad \omega_k(x_1, ..., x_k) = [\omega_i(x_1, ..., x_i), \omega_{k-i}(x_{i+1}, ..., x_k)], \ k > 1.$$

Given G we consider verbal subgroups $\Omega_{\omega_k}(G)$ generated by all values of the word $\omega_k(x_1, ..., x_k)$ in group G.
In [3] V. Sushchansky characterized all verbal subgroups of P_n as terms of the lower central series of P_n.
For characterization of $\Omega_{\omega_k}(P_n)$ we consider subgroups of the form:

$$[\gamma_i(P_n), \gamma_j(P_n)] = \langle[\mathbf{a}, \mathbf{b}], \quad \mathbf{a} \in \gamma i_(P_n), \mathbf{b} \in \gamma_j(P_n)\rangle.$$

Let $s = \lceil \log_p i\rceil \le \lceil \log_p j\rceil = r$, and let

$$\mathbf{a} = [\underbrace{0, ...0}_{s}, a_{s+1}(\underline{x}_s), ..., a_n(\underline{x}_{n-1})] \in \gamma_i(P_n),$$

$$\mathbf{b} = [\underbrace{0, ...0}_{r}, a_{r+1}(\underline{x}_r), ..., a_n(\underline{x}_{n-1})] \in \gamma_j(P_n),$$

where

$$H(a_l(\underline{x}_{l-1})) = p^{l-1} - i + 1 = 1 + \alpha_1 + \alpha_2 \cdot + ... + \alpha_{l-1} \cdot^{l-2},$$
$$H(b_l(\underline{x}_{l-1})) = p^{l-1} - j + 1 = 1 + \beta_1 + \beta_2 \cdot + ... + \beta_{l-1} \cdot^{l-2},$$

for $l \ge s + 1$. We have the following characterization:

Theorem 3. *Let $I = \{l \in \{1, 2, ..., s\} : \ \alpha_l + \beta_l \ge p\}$. For every $i, j \in \mathbb{N}$ we have $[\gamma_i(P_n), \gamma_j(P_n)] = \gamma_k(P_n)$, where*

$$k = i + j + \sum_{l \in I} (\alpha_l + \beta_l - p + 1)\, p^{l-1}.$$

<u>About the proof:</u>

- Induction on n
- Consider the commutators of tables of highest characteristic
- Analyze the height of the polynomial obtained in commutator
- Use the characterization in [2] of normal subgroups of P_n to conclude the statement.

Corollary 4. *Given two outer commutator words ω_k and ω'_k of weight k, the respective verbal subgroups $\Omega_{\omega_k}(P_n)$ and $\Omega_{\omega'_k}(P_n)$ need not coincide.*

References

[1] Bier A., Sushchansky V., „Kaluzhnin's representations of Sylow p-subgroups of automorphism groups of p-adic rooted trees", Algebra Discrete Math. 19 (2015), no. 1, 19–38

[2] Kaloujnine L., „La structure des p-groupes de Sylow des groupes symetriques finis" (French), Ann. Sci. Ecole Norm. Sup. (3) 65, (1948), 239–276

[3] Sushchansky V., „Verbal subgroups of Sylow p-subgroups of finite symmetric groups" (Ukrainian), Visnik Kiev. Univ. 1970, no. 12, 134–141.