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GRIGORCHUK-GUPTA-SIDKI GROUPS AS A SOURCE FOR BEAUVILLE SURFACES

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GROUPS ACTING ON REGULAR ROOTED TREES

Groups acting on regular rooted trees are the subject of intense research over the past few decades after the appearance in the 1980s, when the first Grigorchuk group was defined [2]. This group was designed to be a counterexample to the General Burnside Problem More importantly, it was the first example of a group having intermediate word growth, answering the Milnor Problem.
Later on many different examples and generalizations came into the literature; for example the Gupta-Sidki groups and the second Grigorchuk group. The Grigorchuk-Gupta-Sidki groups (GGS-groups for short) are a family of groups generalizing them.

BEAUVILLE SURFACES

A Beauville surface is a compact complex surface S satisfying the following conditions:

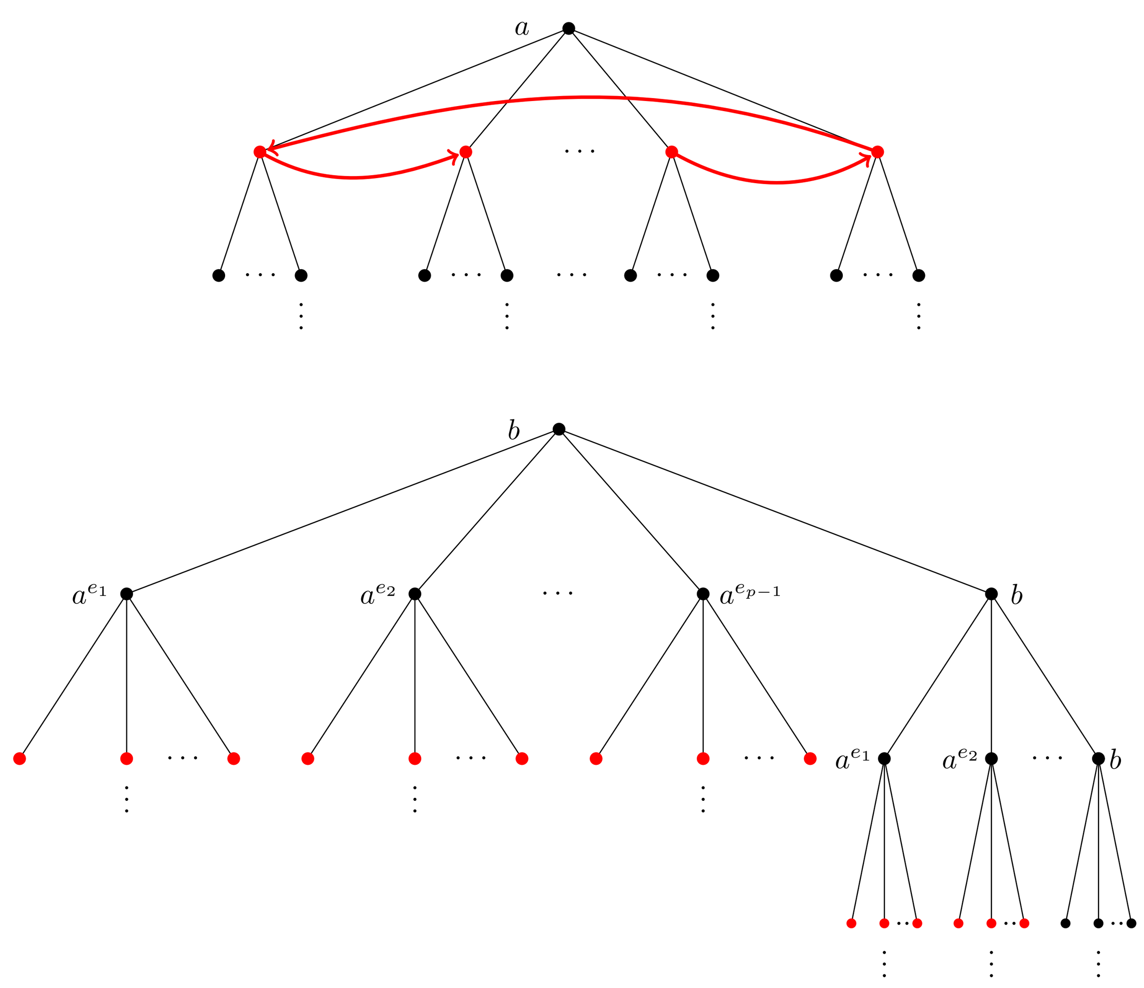
1. $S \cong (C_1 \times C_2)/G$, where C_1 and C_2 are algebraic curves of genera at least 2 and G is a finite group acting freely on $C_1 \times C_2$.
2. G acts faithfully on each curve C_i so that $C_i/G \cong \mathbb{P}_1(\mathbb{C})$ and the covering map $C_i \to C_i/G$ is ramified over three points for $i = 1, 2$.

Then G is called a Beauville group (of unmixed type).
The natural question that arises regarding Beauville surfaces is: which finite groups are Beauville groups?

GRIGORCHUK-GUPTA-SIDKI GROUPS

$\mathcal{T}$= p-adic tree: a tree with root $\emptyset$ such that every vertex has p 'children' for a prime p.
L_n = vertices at distance n from the root.
$\mathrm{Aut}\,\mathcal{T}$ = the group of automorphisms of $\mathcal{T}$.
(an automorphism of $\mathcal{T}$ is a bijection of the vertices that preserves incidence.)
$\mathrm{st}(n) = \{f \in \mathrm{Aut}\,\mathcal{T} \mid f(u) = u \ \ \forall u \in L_n\}$ is the nth level stabilizer.
If $G \leq \mathrm{Aut}\,\mathcal{T}$ then $\mathrm{st}_G(n) = \mathrm{st}(n) \cap G$.
A GGS-group G is a subgroup of $\mathrm{Aut}\,\mathcal{T}$ generated by two automorphisms: a rooted automorphism a permuting the vertices hanging from the root according to the permutation $(1\,2\,\ldots\,p)$, and a recursively defined automorphism b which is defined according to a given vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in \mathbb{F}_p^{p-1}$. A GGS-group is periodic if and only if $e_1 + \cdots + e_{p-1} = 0$ [6].

$G = \langle a, b \rangle$



BEAUVILLE GROUPS

The conditions for a finite group G to be a Beauville group can be reformulated in purely group-theoretical terms as follows.
For a couple of elements $x, y \in G$, we define

$$\Sigma(x, y) = \bigcup_{g \in G} \left(\langle x \rangle^g \cup \langle y \rangle^g \cup \langle xy \rangle^g \right).$$

Then G is a Beauville group if and only if the following conditions hold:

1. G is a 2-generator group.
2. There exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of G such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$.

Then $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are said to form a Beauville structure for G.

In [5], an infinite series of Beauville p-groups was constructed, for every prime p, by considering quotients of ordinary triangle groups. On the other hand, in [3], it was shown that quotients by the terms of the lower p-central series in either the free group of rank 2 or in the free product of two cyclic groups of order p are Beauville groups. In [1], quotients of the Nottingham group over $\mathbb{F}_p$ have been studied in order to construct more infinite families of Beauville p-groups, for an odd prime p.

If G is a GGS-group, then the quotients $G/\mathrm{st}_G(n)$ are finite p-groups generated by two elements. So we can ask whether these quotients are Beauville groups or not. It turns out that the property of being Beauville for these quotients depends on whether G is periodic or not.

MAIN RESULTS [4]

Let p be an odd prime.

PERIODIC CASE:

Theorem A
Let G be a periodic GGS-group over the p-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is a Beauville group if $p \geq 5$ and $n \geq 2$, or $p = 3$ and $n \geq 3$.

NON-PERIODIC CASE:

Theorem B
Let G be a non-periodic GGS-group over the p-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is not a Beauville group for any $n \geq 1$.

Theorem A shows that a periodic GGS-group is a source for the construction of an infinite series of Beauville p-groups. This gives yet another reason why GGS-groups constitute an important family in group theory.

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