On the Schur multiplier of groups

Sumana Hatui

Harish-Chandra Research Institute, Allahabad, India

Abstract: **Let G be a finite p-group and let G' denote its derived subgroup. We denote the Schur multiplier of G by $M(G)$. In 1956, Green proved $|M(G)| \le p^{\frac{1}{2}n(n-1)}$ for p-groups G of order p^n [8]. So $|M(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$, for some $t(G) \ge 0$. It is of interest to characterize the structure of all non-abelian p-groups G by the order of the Schur multiplier $M(G)$, i.e., when $t(G)$ is known. This problem was studied by several authors for various values of $t(G)$. Here we classify non-abelian p-groups G of order p^n for $t(G) = \log_p(|G|) + 1$.**

Later Niroomand improved the Green's bound by proving that for non abelian p-groups G of order p^n with $|G'| = p^k$, $|M(G)| \le p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ [6]. He also classified groups G such that $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ with $k = 1$. Rai classified all finite p-groups of nilpotency class 2 such that $|M(G)|$ attains the bound. We classify finite p-groups of nilpotency class ≥ 3 such that $|M(G)|$ attains the bound [7].

Here we also study about the second cohomology group of central product of groups.

A characterization of finite-p groups by their Schur multiplier

Theorem 0.1. ([1]) *Let G be a finite non-abelian p-group of order p^n with $t(G) = \log_p(|G|) + 1$. Then for odd prime p, G is isomorphic to one of the following groups:*

1. $\Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1\rangle$,
2. $\Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3\alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$,
3. $\Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$
4. $\Phi_2(2111)c = \Phi_2(211)c \times \mathbb{Z}_p$, *where* $\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$,
5. $\Phi_2(2111)d = ES_p(p^3) \times \mathbb{Z}_{p^2}$,
6. $\Phi_3(1^5) = \Phi_3(1^4) \times \mathbb{Z}_p$, *where* $\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1 (i = 1, 2)\rangle$,
7. $\Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1 (i = 1, 2)\rangle$,
8. $\Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^{(p)} = \beta_i^p = 1 (i = 1, 2, 3)\rangle$,
9. $\Phi_{12}(1^6) = ES_p(p^3) \times ES_p(p^3)$,
10. $\Phi_{13}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$,
11. $\Phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = \beta_2^g, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1, 2)\rangle$, *where g is non-quadratic residue modulo p,*
12. $(\mathbb{Z}_p^{(4)} \rtimes \mathbb{Z}_p) \times \mathbb{Z}_p^{(2)}$.
    Moreover for $p = 2$, G is isomorphic to one of the following groups:
13. $\mathbb{Z}_2^{(4)} \rtimes \mathbb{Z}_2$,
14. $G_1 \times \mathbb{Z}_2$, *where* $G_1 = \langle a^4 = b^2 = c^2 = 1, [a, c] = b, [a, b] = [b, c] = 1\rangle$,
15. $G_2 = \langle a^4 = b^4 = c^2 = 1, [a, b] = 1, [a, c] = a^2, [b, c] = b^2\rangle$,
16. $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$,
17. D_{16}, *the Dihedral group of order* 16.

Finite p-groups having Schur multiplier of maximum order

Here we are interested to study about p-groups such that $|M(G)|$ attains the Niroomand's bound.

The following result gives the classification of p-groups G of class 2 such that $|M(G)|$ attains the bound.

Theorem 0.2. ([7]) *Let G be a finite p-group of order p^n and nilpotency class 2 with $|G'| = p^k$. Then $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$ if and only if G is one of the following groups.*
(i) $G_1 = ES_p(p^3) \times \mathbb{Z}_p^{(n-3)}$, *for an odd prime p.*
(ii) $G_2 = \langle \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \mid [\alpha_i, \alpha] = \beta_i, [\alpha_1, \alpha_2] = 1, \alpha^p = \alpha_i^p = \beta_i^p = 1 (i = 1, 2)\rangle$, *for an odd prime p.*
(iii) $G_3 = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^p = \beta_i^p = 1 (i = 1, 2, 3)\rangle$, *for an odd prime p.*

Question: What will happen for groups of nilpotency class ≥ 3?

The next theorem gives the answer of this question:

Theorem 0.3. ([2]) *There is no non-abelian p-group G of order p^n, $p \ne 3$, having nilpotency class $c \ge 3$ with $|G'| = p^k$ and $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$. In particular, $|M(G)| \le p^{\frac{1}{2}(n+k-2)(n-k-1)}$ for p-groups G of nilpotency class $c \ge 3$ and $p \ne 3$.*

Question: what will happen for $p = 3$? Is this above statement true for $p = 3$?

The answer to this question is no. We construct an example, which gives the answer of this question: $G = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \gamma \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, [\beta_3, \alpha_3] = [\beta_2, \alpha_2] = [\beta_1, \alpha_1] = \gamma, \alpha_i^3 = \beta_i^3 = \gamma^3 = 1 (i = 1, 2, 3)\rangle$ of order 3^7. Using HAP of GAP we see that $|M(G)| = p^{\frac{1}{2}(n+k-2)(n-k-1)+1} = p^{10}$.

Question: Does there exist finite p-groups of arbitrary nilpotency class for which this new bound is attained? The answer of this question is yes for nilpotency class 3 and 4, see the following examples:

Example1: $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha, \alpha_1] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_2, \alpha_1] = \alpha_4, \alpha^p = \alpha_i^p = 1 (i = 1, 2, 3, 4)\rangle$. This is group of order p^5 with $|G'| = p^3$. Nilpotency class of G is 3. For $p = 5, 7, 11, 13, 17$ using HAP of GAP we obtain $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Note that $|M(G)| = p^{\frac{1}{2}(5+3-2)(5-3-1)} = p^3$.

Example2: Consider the group $G = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 (i = 1, 2, 3)\rangle$. This is group of order p^5 with $|G'| = p^3$. Nilpotency class of G is 4. For $p = 5, 7, 11, 13, 17$ using HAP of GAP we obtain $M(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Note that $|M(G)| = p^{\frac{1}{2}(5+3-2)(5-3-1)} = p^3$.

Open Question: Classify p-groups such that $|M(G)|$ attains the new bound?

The Schur multiplier of central product of groups

Let a finite group G be the direct product of two groups H and K. Then the formulation of the Schur multiplier of G in terms of the Schur multipliers of H and K was given by Schur himself. Such a formulation, when G is a semidirect product of groups H and K, was given by Tahara. Let G be a central product of two groups H and K. We study second cohomology group of G, having coefficients in a divisible abelian group D with trivial G-action, in terms of the second cohomology groups of certain quotients of H and K.

Theorem 0.4. ([4]) *Let H, K be finite groups, let U, V be isomorphic central subgroups of H, K respectively, and let ϕ be an isomorphism from U onto V. Then the multiplicator of the central product G of H and K amalgamating U with V according to ϕ contains a subgroup isomorphic with $H/U \otimes K/V$.*

Theorem 0.5. ([5]) *Let W be central in $A = H \times K$ with quotient G. Let U and V be the images of W under the projection of A onto H and K respectively. Then $H/U \otimes K/V$ is a quotient of $H_2(G, \mathbb{Z})$.*

Set $Z = H' \cap K'$. The following result provides a reduction to the case when $Z = 1$.

Theorem 0.6. ([3]) *Let B be a subgroup of G such that $B \le Z$. Then* $\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/B, D)/N$, *where* $N \cong \mathrm{Hom}(B, D)$.

Now we prove the folloing theorem:

Theorem 0.7. ([3]) *Let* $L \cong \mathrm{Hom}\big((A \cap H')/Z, D\big)$, $M \cong \mathrm{Hom}\big((A \cap K')/Z, D\big)$ *and* $N \cong \mathrm{Hom}(Z, D)$. *Then the following statements hold true:*
(i) $\big(\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M\big)/N \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ *embeds in* $\mathrm{H}^2(G, D)$.
(ii) $\mathrm{H}^2(G, D)$ *embeds in* $\big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/N \oplus \mathrm{Hom}(H \otimes K, D)$.

In particular, for $D = \mathbb{C}^\times$, assertion (i) of Theorem 0.7 provides a refinement of Theorem 0.4 and Theorem 0.5.

Corollary 0.8. *(i) If $A = Z$, then*

$$\mathrm{H}^2(G, D) \cong \big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/\mathrm{Hom}(Z, D) \oplus \mathrm{Hom}(H/Z \otimes K/Z, D).$$

(ii) If $\inf : \mathrm{H}^2(H/A, D) \to \mathrm{H}^2(H/Z, D)$ *and* $\inf : \mathrm{H}^2(K/A, D) \to \mathrm{H}^2(K/Z, D)$ *are epimorphisms, then*

$$\mathrm{H}^2(G, D) \cong \big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/\mathrm{Hom}(Z, D) \oplus \mathrm{Hom}(H/A \otimes K/A, D).$$

More precisely, the first embedding in Theorem 0.7 is an isomorphism.

Theorem 0.9. *If the second embedding in Theorem 0.7 is an isomorphism, then so is the first.*

• We have examples (all of them are finite p-groups) to show that the following three situations of Theorem 0.7 can indeed occur.
(i) The first embedding in Theorem 0.7 can very well be an isomorphisms, but the second one can still be strict (i.e., not an isomorphism).

(ii) Neither of the two embeddings of Theorem 0.7 is necessarily an isomorphism.

(iii) Both the embeddings in Theorem 0.7 can be isomorphisms.

References

1. Sumana Hatui, *A characterization of finite p-groups by their Schur multiplier*, Proc. Indian Acad. Sci. Sect. A Math. Sci., To appear.
2. Sumana Hatui, *Finite p-groups having Schur multiplier of maximum order*, Journal of Algebra., **472** (2014), no. 1, 490-497.
3. Sumana Hatui, L. R. Vermani, Manoj K. Yadav, *The Schur multiplier of central product of groups*, Journal of Pure and Applied Algebra, To appear.
4. Wiegold, J., *Some groups with non-trivial multiplicators*, Math. Z. **120** (1971), 307-308.
5. Eckmann, B., Hilton, P. J. and Stammbach, U. *On the Schur multiplicator of a central quotient of a direct product of groups,* J. Pure Appl. Algebra **3** (1973), 73-82.
6. Niroomand, P, *On the order of Schur multiplier of non-abelian p groups*, Journal of Algebra, **322**, (2009), 4479-4482.
7. Rai, Pradeep K. *On classification of groups having Schur multiplier of maximum order*, Arch. Math., (2016).
8. Green, J.A., *On the number of automorphisms of a finite group*, Proc. Roy. Soc, A **237**, (1956), 574-581.

Acknowledgement:

I am thankful to the organizers for giving me opportunity to present my work. I also wish to thank Department of atomic energy, Government of India and Infosys grant for providing me travel support.