On finite p-groups of conjugate rank 1

Tushar Kanta Naik

Introduction

A finite group G is said to be of conjugate type $\{1,\ m\}$, if for all $g \in G \setminus Z(G)$, $[G : C_G(g)] = m$. Here, we also say G is of conjugate rank 1.

- N. Ito[1] proved that if G is of conjugate type $\{1, m\}$, then G is nilpotent and $m = p^k$, for some prime p and integer $k \geq 1$.
- K.Ishikawa [2] proved that if G be a p-group with conjugate type $\{1, p^n\}$, then nilpotency class of G is exactly 2, when $p = 2$ and at most 3, when $p > 2$
- K.Ishikawa [3] also classified finite p-groups of conjugate type $\{1, p\}$and $\{1, p^2\}$ upto isoclinism.
- Two finite groups G and H are said to be ***isoclinic*** if there exists an isomorphism ϕ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism θ of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \xrightarrow{a_G} & \gamma_2(G) \\ {\scriptstyle \phi\times\phi}\downarrow & & \downarrow{\scriptstyle \theta} \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & \gamma_2(H). \end{array}$$

Finite p-groups of conjugate type $(1, p^n)$ and nilpotency class 3

Theorem 3: Let $p > 2$ be a prime and $n \geq 1$ an integer. Then there exist finite p-groups of nilpotency class 3 and conjugate type $(1, p^n)$ if and only if n is even. For each positive even integer $n = 2m$, every finite p-group of nilpotency class 3 and of conjugate type $(1, p^{2m})$ is isoclinic to the group $G_m/Z(G_m)$, where G_m is as

$$G_m = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ c & b & 1 & 0 & 0 \\ d & ab-c & a & 1 & 0 \\ f & e & c & b & 1 \end{bmatrix} : a, b, c, d, e, f \in \mathbb{F}_{p^m} \right\}.$$

Finite p-groups of conjugate type $(1, p^3)$

Theorem 1: Let G be a finite p-group of conjugate type $\{1, p^3\}$, $p > 2$. Then G is of class 2 and isoclinic to one of following groups:

1. Finite Camina p-group K with $|\gamma_2(K)| = p^3$.
2. The group H_1, defined as
$$H_1 = \langle a_1, a_2, a_3, a_4, b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}; [a_i, a_j] = b_{ij}, a_i^p = a_4^p = b_{ij}^p = 1; (1 \leq i < j \leq 4)\rangle.$$
3. The quotient group H_1/M_1, where M_1 is a central subgroup of H_1, presented as $M_1 = \langle [a_1,\ a_2][a_3,\ a_4]\rangle$.
4. The quotient group H_1/N_1, where H_1 is a central subgroup of H_1, presented as $N_1 = \langle [a_1,\ a_2][a_3,\ a_4],\ [a_1,\ a_3][a_2,\ a_4]^t\rangle$, with t any fixed non-square modulo p.

Set $\hat{H} := \{H : H$ is freest 2-group with 4 generators satisfying $exp(H) = 4$, $|\ H\ | = 2^{10}$, $Z(H) = H' \cong$ elementary abelian 2-group of order $2^6\}$. For simplicity of notation, we assume that a group $H \in \hat{H}$ is generated by a, b, c and d.

Theorem 2: Let G be a finite 2-group of conjugate type $\{1, 8\}$. Then G is isoclinic to one of following groups:

1. Finite Camina 2-group L with $|\gamma_2(L)| = 8$.
2. Any fixed group $H_2 \in \hat{H}$.
3. The quotient group H_2/M_2, where M_2 is a central subgroup of H_2 given by $M_2 = \langle [a,\ b][c,\ d]\rangle$.
4. The quotient group H_2/N_2, where N_2 is a central subgroup of H_2 given by $M_2 = \langle [a,\ b][c,\ d],\ [a,\ c][b,\ d][c,\ d]\rangle$.

Probability Distribution Associated To Commutator Word Map

Let G be a finite group and $g \in K(G)$. Here $K(G)$ denotes the set of commutators of G. Here, we define;

$$Pr_g(G) = |\{(x, y) \mid [x,\ y] = g\}|/|G|^2.$$

$$P(G) = \{Pr_g(G) \mid 1 \neq g \in K(G)\}.$$

Theorem 4: Let $n \geq 1$ be a given positive integer. Then there always exist a group G (depending on n) of nilpotency class 2 and conjugate type $(1, p^m)$ such that $|P(G)| = n$. In particular, if we take

$$G = \langle a_1, \ldots, a_r \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, a_i^p = a_r^p = b_{ij}^p = 1, 1 \leq i < j \leq r, 1 \leq k \leq r$$

with $r = n^2 + n - 2$ and

$$H = \langle [a_1, a_2][a_3, a_4], [a_5, a_6][a_7, a_8], [a_5, a_6][a_9, a_{10}], [a_{11}, a_{12}][a_{13}, a_{14}], [a_{11}, a_{12}][a_{15}, a_{16}], [a_{11}, a_{12}][a_{17}, a_{18}], \vdots [a_{\alpha+1}, a_{\alpha+2}][a_{\alpha+3}, a_{\alpha+4}], [a_{\alpha+1}, a_{\alpha+2}][a_{\alpha+5}, a_{\alpha+6}] \ldots [a_{\alpha+1}, a_{\alpha+2}][a_{\alpha+2n-1}, a_{\alpha+2n}]\rangle;$$

where $\alpha = (n-2)(n+1)$. Then $|P(G/H)| = n$

Theorem 5: Let G be a finite p-group of conjugate type $(1, p^{2n})$ and nilpotency class 3. Then for $g \in G'$,

$$Pr_g(G) = \begin{cases} \dfrac{p^{3n} + p^{2n} - 1}{p^{5n}}, & \text{if } g = 1 \\ \dfrac{p^{2n} - 1}{p^{5n}}, & \text{if } 1 \neq g \in G'. \end{cases}$$

Hence $|P(G)| = 1$.

References

[1] N. Ito. On finite groups with given conjugate types I. *Nayoga Math. J.*, 6:17-28, 1953.

[2] K. Ishikawa. Finite p-groups up to isoclinism, which have only two conjugacy lengths. *J. Algebra* 220: 333-345, 1999.

[3] K. Ishikawa. On finite p-groups which have only two conjugacy lengths. *Israel J. Math.* 129: 119-123, 2002.

[4] T. K. Naik, *On the probability distribution associated to commutator word map in finite groups II, submitted.*

[5] T. K. Naik and M. K. Yadav, Finite p-groups of conjugate type $\{1, p^3\}$*, J. Group Theory,* **21(1)***, 65-82 (2018).*

[6] T. K. Naik, R. D. Kitture, M. K. Yadav, Finite p-Groups of Nilpotency Class 3 with Two Conjugacy Class Sizes, submitted.

Harish-Chandra Research Institute, Jhunsi, Allahbad, India-211019

HBNI, Training School Complex, Anushaktinagar, Mumbai, India-400094