A crystallographic cocompact hyperbolic Coxeter group $\mathfrak{W}$

A model in $O_4(\mathbb{R}, \mathfrak{Q})$

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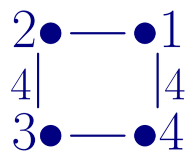
Abstract

Through the geometric representation, an hyperbolic Coxeter group can be thought as reflection group acting on hyperbolic space. The presented Coxeter $\mathfrak{W}$ with Coxeter graph



(1)

is a cocompact, crystallographic and hyperbolic group. The interest for $\mathfrak{W}$ is due to the link between a cocompact crystallographic hyperbolic Kac-Moody Lie algebra and in particular with *Kac' denominator formula* to calculate the roots and their multiplicities. From litterature we know that $\mathfrak{W}$ is conjugate to a group, that is commensurable with the projective orthogonal group $PO_4(\mathbb{Z}, \mathfrak{Q})$ respect to the quadratic form $\mathfrak{Q}(x_0, x_1, x_2, x_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$. Therefore, the aim of my resarch is to show a model of $\mathfrak{W}$ inside the orthogonal group $O_4(\mathbb{R}, \mathfrak{Q})$.

Introduction to Coxeter groups

A Coxeter group is an abstract group with the following presentation

$$\langle s_1, \dots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2$ for $i \neq j$. If no relations occur between s_i and s_j, then $m_{ij} = \infty$.

The geometric representation of a Coxeter group W is a group homomorphism $\phi : W \to GL(V)$, sending s_i to a reflection $s_{\alpha_i}(v) := v - 2\frac{B(v,\alpha_i)}{B(\alpha_i,\alpha_i)}\alpha_i$, where B is a bilinear form on V preserved by the group $\phi(W)$ and $v, \alpha_i \in V$.

We can reassume the presentation of the Coxeter group with the Coxeter graph, that is a finite graph whose vertices are integers $i = 1, \dots, n$ associated to every element s_i and edges are labelled with integers $m_{ij} \geq 3$, except when $m_{ij} = 3$; instead if $m_{ij} = 2$ for $i \neq j$, the vertices i and j are not joined by edges. $\mathfrak{W}$ has associated the Coxeter graph (1).

Why interest for $\mathfrak{W}$?

The interest for $\mathfrak{W}$ comes from the link with the cocompact hyperbolic Kac-Moody Lie algebra $\mathfrak{L}$ of rank 4 with Generalised Cartan Matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ -2 & 0 & -1 & 2 \end{bmatrix}.$$

The group $\mathfrak{W}$ as reflection group is the Weyl group of $\mathfrak{L}$. Open problem is to calculate the imaginary roots $\mathfrak{L}$ and their multeplicity, looking for a way to parameterize the **Kac' denominator formula**:

$$\sum_{w \in \mathfrak{W}} (-1)^{l(w)} e(c(w)) = \prod_{\alpha \in \Omega^+} (1 - e(\alpha))^{m_\alpha}$$

where

(i) Ω^+ are the positive roots;

(ii) $e(\alpha)$ is a formal exponential;

(iii) $l(w)$ is the littles integer r such that $w = s_1 \dots s_r$ with s_i generators of $\mathfrak{W}$;

(iv) $c : \mathfrak{W} \to \Lambda^+$ is a co-cycle that sends every elements of $\mathfrak{W}$ in the sum of the positive roots that the element sends in the negative ones.

From literature

The **exceptional isomorphism** $\Psi : SL(2, \mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q})$ together with the link between the theory of **quaternion algebra** and with the **discrete, cocompact subgroups of** $SL(2, \mathbb{C})$ will be useful tools to construct a model for $\mathfrak{W}$.

(a) $\mathfrak{W}$ is a crystallographic, cocompact and hyperbolic Coxeter group.

(b) $\mathfrak{W}$ is conjugate to a group commensurable with $PO_4(\mathbb{Z}, \mathfrak{Q})$, with $\mathfrak{Q}(x_0, x_1, x_2, x_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$ quadratic form.

(c) There exists an exceptional group isomorphism $\Psi : SL(2, \mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q})$. $Im\Psi = SO_4^+(\mathbb{R}, \mathfrak{Q})$ consists of those elements in $SO_4(\mathbb{R}, \mathfrak{Q})$ which have a positive entry in the left upper corner.

(d) Let $\mathcal{A} = \left(\frac{a,b}{\mathbb{K}}\right)$ be the Hilbert symbol to indicate a quaternion algebra and $\mathcal{A}^1$ be the group of elements of $\mathcal{A}$ of norm 1. There exists an injective-group homomorphism $\Phi : \mathcal{A}^1 \to SL(2, \mathbb{C})$.

(e) Let $\mathcal{A}$ as in the previous point, for an order $\mathcal{R} \subseteq \mathcal{A}$, the group $\Gamma := \Phi(\mathcal{R}^1)$ is a discrete subgroup of $SL(2, \mathbb{C})$ and in particular Γ is cocompact if and only if $\mathcal{A}$ is a skew field.

Results

(A) The quaternion algebra associated to $\mathfrak{W}$ is $\mathcal{S} = \left(\frac{-1,-1}{\mathbb{Q}(i\sqrt{7})}\right)$. $\mathcal{S}$ is a skew-field over $\mathbb{Q}(i\sqrt{7})$, therefore $\Phi(\mathbf{S}^1)$ is a cocompact subgroup of $SL(2, \mathbb{C})$.

(B) $\Phi(\mathcal{S}^1) = \{ \begin{pmatrix} x_0 + x_1\sqrt{-1} & x_2\sqrt{-1} + x_3 \\ x_2\sqrt{-1} - x_3 & x_0 - x_1\sqrt{-1} \end{pmatrix} \mid x_i \in \mathbb{Q}(i\sqrt{7}) \} \subseteq SL(2, \mathbb{Q}(i, \sqrt{7})$.

(C) The exceptional isomorphism $\Psi : SL(2, \mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q})$ is a group isomorphism, because we can consider $\tilde{\Psi} : \langle \sigma \rangle \ltimes SL(2, \mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q})$, where σ is an involution that acts by conjugation sending every coefficient of the matrix of $SL(2, \mathbb{C})$ in its conjugate in $\mathbb{C}$. Moreover the image of σ in $O_4(\mathbb{R}, \mathfrak{Q})$ is a reflection.

(D) The generators of the group $\mathfrak{W}$ in a **commensurable** group of $\langle \sigma \rangle \ltimes SL(2, \mathbb{C})$ are given by the formula

$$s_j = \frac{1}{\sqrt{D_j}} \sigma \begin{pmatrix} P_{4j} + P_{3j} i & -P_{2j} i + P_{1j}\sqrt{7} i \\ -P_{2j} i - P_{1j}\sqrt{7} i & P_{4j} - P_{3j} i \end{pmatrix}$$

where (P_{ij}) is the change of basis matrix from the basis of simple roots to the standard basis for a quadratic space $(V; \mathfrak{Q})$ and D_j is the norm of the root α_j.

(E) $\tilde{\Psi}(s_j) = S_j$ are the generators of $\mathfrak{W}$ in the lattice $O_4(\mathbb{Z}, \mathfrak{Q})$.

(F) Let v_0, v_1, v_2, v_3 be the standard basis for a quadratic space $(V; \mathfrak{Q})$. The $\mathbb{C}$-linear map $\mu : V_{\mathbb{Z}} \to M(2, \mathbb{C})$, $\mu(av_1 + bv_2 + cv_3 + dv_4) = \begin{pmatrix} -b - ci & a - \sqrt{7}d \\ a + \sqrt{7}d & b - ci \end{pmatrix}$ such that $B(\alpha, \alpha) = -det(\mu(\alpha))$, give us a model for the root system in $M(2, \mathbb{C})$.

(G) The acton of the group $\mathfrak{W}$ on $\mu(V_{\mathbb{Z}})$ is $(\sigma A_{i_1} \dots \sigma A_{i_k}) \cdot X = \sigma A_{i_1} \dots \sigma A_{i_k} X A_{i_k} \sigma \dots A_{i_1} \sigma$. Then the map μ is $\mathfrak{W}$-equivariant. One can identify $V_{\mathbb{Z}}$ with $\mu(V_{\mathbb{Z}})$.

(H) $\mu(\Phi_R) = \{X \in \mu(V_{\mathbb{Z}}) \mid det(X) = -2, -4\}$ is the set of real roots.
$\mu(\Phi_I) = \{X \in \mu(V_{\mathbb{Z}}) \mid det(X) \geq 0\}$ is the set of imaginary roots.

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